29. Output Signal-to-Noise Ratios in AM and FM

In digital modulation systems, we were able to determine the system performance quite uniquely by calculating the probability of error. Because of the continuous nature of analogue modulation systems, it is difficult to adopt this approach. Instead, we shall determine and compare the performance of analogue modulation systems on the basis of signal-to-noise ratio (*SNR*) at the receiver input and output.

Performance of Amplitude Modulation [1, 2]

We have seen that the a normal amplitude-modulated signal is given by

$$s_c(t) = [A + m(t)] \cos 2\pi f_c t \tag{29.1}$$

$$= A[1 + \frac{1}{A} m(t)] \cos 2\pi f_c t \tag{29.2}$$

where A is a constant, $m(t)$ is the modulating signal, and f_c is the carrier frequency. The *modulation index* is defined as [1]

$$m = \frac{|\min \ m(t)|}{A} \tag{29.3}$$

and equation (29.2) can be written as

$$s_c(t) = A[1 + \frac{m}{|\min \ m(t)|} m(t)] \cos 2\pi f_c t \tag{29.4}$$

$$= A[1 + Km(t)] \cos 2\pi f_c t \tag{29.5}$$

where $K = m / |\min \ m(t)|$.

Consider the amplitude demodulator shown in Figure 29.1.

Figure 29.1 Idealised amplitude demodulation using an envelope detector.

In the presence of additive noise, the signal plus narrowband noise at the output of the band-pass filter is

$$v(t) = A[1 + Km(t)] \cos 2\pi f_c t + n(t) \tag{29.6}$$

where

$$n(t) = x(t) \cos 2\pi f_c t - y(t) \sin 2\pi f_c t \tag{29.7}$$

The input signal power to the envelope detector is

$$S_i = \frac{A^2}{2} + \frac{A^2 K^2 P_m}{2} \tag{29.8}$$

where P_m is the average power of $m(t)$. Substituting equation (29.7) into equation (29.6), we get

$$\begin{aligned} v(t) &= \{A[1 + Km(t)] + x(t)\}\cos 2\pi f_c t - y(t) \sin 2\pi f_c t \\ &= \sqrt{\{A[1+Km(t)]+x(t)\}^2 + y(t)^2}\ \cos [2\pi f_c t + \phi(t)] && (29.9) \\ &= \sqrt{\{A[1+Km(t)]+x(t)\}^2 + y(t)^2}\ \cos\theta(t) && (29.10) \\ &= \mathrm{Re}\ \{ \sqrt{\{A[1+Km(t)]+x(t)\}^2 + y(t)^2}\ e^{j\theta(t)}\} && (29.11) \end{aligned}$$

where

$$\phi(t) = \tan^{-1} \frac{y(t)}{A[1+Km(t)]+x(t)} \tag{29.12}$$

and

$$\theta(t) = 2\pi f_c t + \phi(t) \tag{29.13}$$

Figure 29.2 shows the signal $v(t)$ in polar form with an envelope of $\sqrt{\{A[1+Km(t)]+x(t)\}^2 + y(t)^2}$.

Figure 29.2 Phasor diagram for AM signals plus narrowband noise.

For *large input signal-to-noise ratio*, the output of the low-pass filter is

$$e(t) \approx A[1 + Km(t)] + x(t) \tag{29.14}$$

Ignoring the first term arising from the carrier signal, the output signal power is

$$S_o = \frac{A^2 K^2 P_m}{2} \tag{29.15}$$

Let $G_n(f)$ be the power spectral density of the narrowband noise $n(t)$ shown in Figure 29.3.

Figure 29.3 Narrowband noise power spectral density.

The mean noise power entering the envelope detector is

$$
\begin{aligned}
E[n(t)^2] = N_i &= \int_{-\infty}^{\infty} G_n(f)\, df \\
&= \int_{-f_c-B}^{-f_c+B} \frac{n_0}{2}\, df + \int_{f_c-B}^{f_c+B} \frac{n_0}{2}\, df \\
&= n_0 B + n_0 B \\
&= 2 n_0 B \qquad (29.16)
\end{aligned}
$$

where B is the bandwidth of the modulating signal $m(t)$.

Also, let $G_x(f)$ and $G_y(f)$ be the power spectral densities of the quadrature components $x(t)$ and $y(t)$ of the noise $n(t)$. They are found to be given by

$$
\begin{aligned}
G_x(f) &= G_y(f) \\
&= 2\, G_n(f + f_c) \qquad (29.17) \\
&= n_0 \qquad (29.18)
\end{aligned}
$$

The mean noise power at the output of the low-pass filter is

$$
\begin{aligned}
N_o &= \int_{-B}^{B} G_x(f)\, df \\
&= 2 n_0 B \qquad (29.19)
\end{aligned}
$$

Therefore the output signal-to-noise ratio is

$$
\frac{S_o}{N_o} = \frac{K^2 P_m}{1 + K^2 P_m} \frac{S_i}{N_i} \qquad (29.20)
$$

If $m(t) = a_m \cos 2\pi f_c t$, then $P_m = \frac{a_m^2}{2}$, $K = m / a_m$, and

$$\frac{S_o}{N_o} = \frac{m^2}{2 + m^2} \frac{S_i}{N_i} \tag{29.21}$$

For large *input SNR* and a fixed modulation index, the output *SNR* is directly proportional to the input *SNR*. When the input *SNR* << 1, the envelope signal is primarily dominated by the envelope of the noise signal and the modulating signal is badly mutilated. Under this circumstance, it is meaningless to talk about output *SNR*. The loss of the modulating signal at low input *SNR* is called the *threshold effect*. The threshold occurs when the input *SNR* < 10 dB. It can be shown that, for a small input *SNR*, the output *SNR* of the envelope detector is proportional to the squared input *SNR* [2, 3]. Figure 29.4 shows a typical asymptotic signal-to-noise characteristic for an envelope detector.

Figure 29.4 Asymptotic signal-to-noise characteristic for envelope detector.

Performance of Frequency Modulation [1, 3, 4]

Let k_f be a constant and $m_f(t)$ be the modulating signal. A frequency-modulated signal is given by [1]

$$s_c(t) = A\cos\,[2\pi f_c t + \phi(t)] \tag{29.22}$$

$$= A \cos\, \theta(t) \tag{29.23}$$

where

$$\theta(t) = 2\pi f_c t + \phi(t) \tag{29.24}$$

$$\phi(t) = k_f \int_{-\infty}^{t_1} m_f(t)dt \tag{29.25}$$

and

$$\frac{d\phi(t)}{dt} = k_f m_f(t) \tag{29.26}$$

The *peak frequency deviation* is defined as [4]

$$\Delta f = \max \left| \frac{1}{2\pi} \frac{d\phi(t)}{dt} \right| \tag{29.27}$$

Because of the difficulty of analysing general frequency-modulated signals, we shall only consider a sinusoidal modulating signal. Let the modulating signal of a single-tone FM signal be

$$m_f(t) = a_m \cos 2\pi f_m t \tag{29.28}$$

Substituting (29.28) into (29.25), we have

$$\phi(t) = \frac{k_f a_m}{2\pi f_m} \sin 2\pi f_m t \tag{29.29}$$

$$= \beta_f \sin 2\pi f_m t \tag{29.30}$$

where $\beta_f = k_f a_m/(2\pi f_m)$, and the *frequency modulation index* β_f is only defined for a sinusoidal modulating signal. Substituting equation (29.30) into (29.22), we have

$$s_c(t) = A\cos(2\pi f_c t + \beta_f \sin 2\pi f_m t) \tag{29.31}$$

The input signal power to the frequency demodulator is

$$S_i = \frac{A^2}{2} \tag{29.32}$$

and

$$\phi(t) = k_f \int_{-\infty}^{t_1} m_f(t)dt = \beta_f \sin 2\pi f_m t \tag{29.33}$$

Differentiating both sides with respect to time and solving for $m_f(t)$, we get

$$m_f(t) = \frac{2\pi f_m \beta_f}{k_f} \cos 2\pi f_m t$$

Hence the output signal power is

$$S_o = \frac{1}{2}\left(\frac{2\pi B \beta_f}{k_f}\right)^2 \tag{29.34}$$

where $B = f_m$ is the bandwidth of the modulating signal $m_f(t)$.

To simplify the analysis of noise in FM systems, we assume $m_f(t) = 0$ and consider the frequency demodulator shown in Figure 29.5.

Figure 29.5 Frequency demodulation using a frequency discriminator.

The input signal to the limiter is

$$r(t) = A\cos 2\pi f_c t + n(t) \tag{29.35}$$

where

$$n(t) = x(t) \cos 2\pi f_c t - y(t) \sin 2\pi f_c t \tag{29.36}$$

Substituting $n(t)$ into equations (29.35), we get

$$\begin{aligned} r(t) &= [x(t) + A]\cos 2\pi f_c t - y(t) \sin 2\pi f_c t \\ &= \sqrt{[x(t)+A]^2 + y(t)^2}\ \cos\,[2\pi f_c t + \phi(t)] && (29.37) \\ &= \sqrt{[x(t)+A]^2 + y(t)^2}\ \cos\theta(t) && (29.38) \end{aligned}$$

where

$$\phi(t) = \tan^{-1} \frac{y(t)}{x(t)+A} \tag{29.39}$$

and

$$\theta(t) = 2\pi f_c t + \phi(t)$$

The signal at the output of the limiter is

$$v(t) = K\cos\theta(t) \tag{29.40}$$

Setting $K = 1$ and taking the derivative of $\theta(t)$, the output of the differentiator is given by

$$v'(t) = -\sin\theta(t)\,\frac{d\theta(t)}{dt} \tag{29.41}$$

$$= -\sin\theta(t)\ [2\pi f_C + \frac{d\phi(t)}{dt}]$$

and the signal at the output of the low-pass filter is

$$e(t) = 2\pi f_C + \frac{d\phi(t)}{dt}$$

where

$$\frac{d\phi(t)}{dt} = \dot{\phi} = \frac{[x(t)+A]\frac{dy(t)}{dt} - y(t)\frac{dx(t)}{dt}}{y(t)^2 + [x(t)+A]^2} \qquad (29.42)$$

For large signal-to-noise ratio at the input of the frequency demodulator,

$$\dot{\phi} \approx \frac{1}{A}\frac{dy(t)}{dt} \qquad (29.43)$$

and the signal at the output of the low-pass filter is $2\pi f_C + \frac{1}{A}\frac{dy(t)}{dt}$. We can ignore the first term arising from the carrier frequency, which can be removed by a blocking capacitor. Thus, $y(t)$ must be the noise signal at the input of the differentiator with a transfer function

$$H(f) = \frac{2\pi f}{A}j$$

Proof.

Taking the Fourier transform of $\frac{d\phi(t)}{dt} = \frac{1}{A}\frac{dy(t)}{dt}$, we get $F[\frac{d\phi(t)}{dt}] = \frac{2\pi f}{A}j\ Y(f) = H(f)Y(f)$ where $H(f) = \frac{2\pi f}{A}j$. Q.E.D.

The transfer function of the differentiator in an FM receiver is shown in Figure 29.6.

Figure 29.6 Transfer function of a differentiator in an FM receiver.

Let $G_n(f)$ be the power spectral density of the narrowband noise $n(t)$ shown in Figure 29.7 (a).

Figure 29.7 Narrowband noise power spectral density.

The mean noise power entering the differentiator is

$$
\begin{aligned}
E[n(t)^2] = N_i &= \int_{-\infty}^{\infty} G_n(f)\, df \\
&= \int_{-\frac{B_T}{2}-f_c}^{-f_c+\frac{B_T}{2}} \frac{n_0}{2}\, df + \int_{f_c-\frac{B_T}{2}}^{\frac{B_T}{2}+f_c} \frac{n_0}{2}\, df \\
&= n_0 B_T + n_0 B_T \\
&= 2 n_0 B_T \qquad (29.44)
\end{aligned}
$$

Let $G_x(f)$ and $G_y(f)$ be the power spectral densities of the quadrature components $x(t)$ and $y(t)$ of the noise $n(t)$. The noise power spectral density at the output of the differentiator is given by

$$
\begin{aligned}
G_{\dot{\phi}}(f) &= |H(f)|^2 G_y(f) \\
&= \left[\frac{2\pi f}{A}\right]^2 G_y(f) \\
&= 2\left[\frac{2\pi f}{A}\right]^2 G_n(f+f_c) \\
&= \left[\frac{2\pi f}{A}\right]^2 n_0
\end{aligned}
$$

This is shown in Figure 29.7 (b).

The mean noise at the output of the low-pass filter with a bandwidth of B Hz is

$$
\begin{aligned}
N_o &= \int_{-B}^{B} G_{\dot{\phi}}(f)\, df = \frac{2(2\pi)^2 n_0}{3A^2} B^3 \\
&= \frac{2(2\pi)^2 n_0}{3A^2} B^3 \qquad (29.45)
\end{aligned}
$$

Therefore the output signal-to-noise ratio is

$$\frac{So}{No} = \frac{1}{2}\left(\frac{2\pi B\beta_f}{k_f}\right)^2 \frac{3A^2}{2(2\pi)^2 n_o B^3} = \frac{3}{(k_f)^2}\beta_f^2 \frac{A^2}{2N_i} = \frac{3}{(k_f)^2}\beta_f^2 \frac{S_i}{N_i} \tag{29.46}$$

As β increases, the bandwidth increases and the output SNR increases. For a fixed input SNR, an improvement in output SNR is possible with FM systems. Can we keep improving the output SNR by simply increasing β? If the modulating signal bandwidth and the carrier power are fixed, more noise must be accepted by the limiter when we increase β. Eventually, the noise power becomes comparable with the carrier signal power. Equation (29.46) does not hold anymore and the noise is found to take over the system. A so called *threshold effect* occurs at a certain input SNR. For angle-modulated systems it is common to call the input SNR the *carrier-to-noise ratio (CNR)*. Figure 29.8 shows a typical signal-to-noise characteristic for a frequency discriminator.

Figure 29.8 Signal-to-noise characteristic for frequency discriminator.

To avoid the threshold effect, the CNR > 10 dB and β > 1/3 for output $SNR > CNR$. It should be noted that we cannot improve the output SNR of narrowband FM systems ($\beta << \pi/2$).

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[2] S. Haykin, Communication Systems, 4/e, John Wiley & Sons, 2001.

[3] M. Schwartz, Information Transmission, Modulation, and Noise, 4/e, McGraw-Hill, 1990.

[4] L. W. Couch II, Digital and Analog Communication Systems, 5/e, Prentice Hall, 1997.

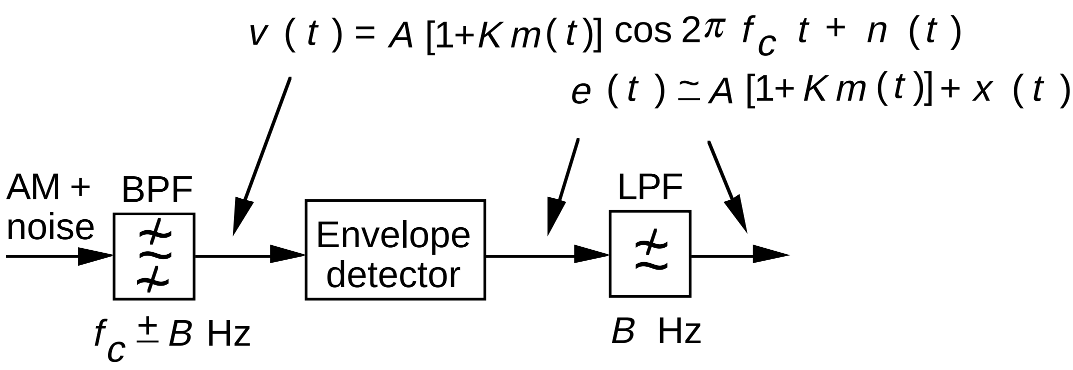

Figure 29.1 Idealised amplitude demodulation using an envelope detector.

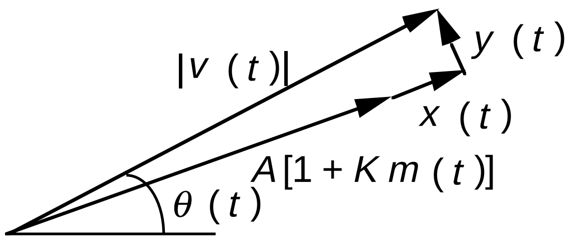

Figure 29.2 Phasor diagram for AM signals plus narrowband noise.

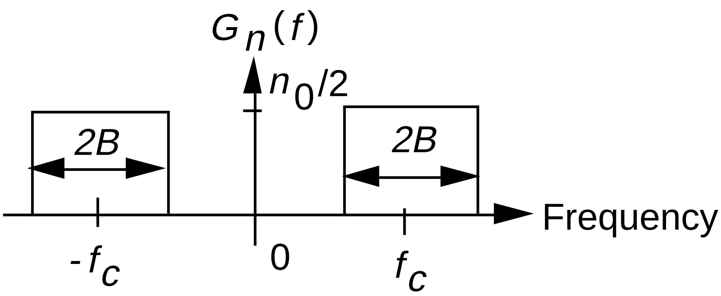

Figure 29.3 Narrowband noise power spectral density.

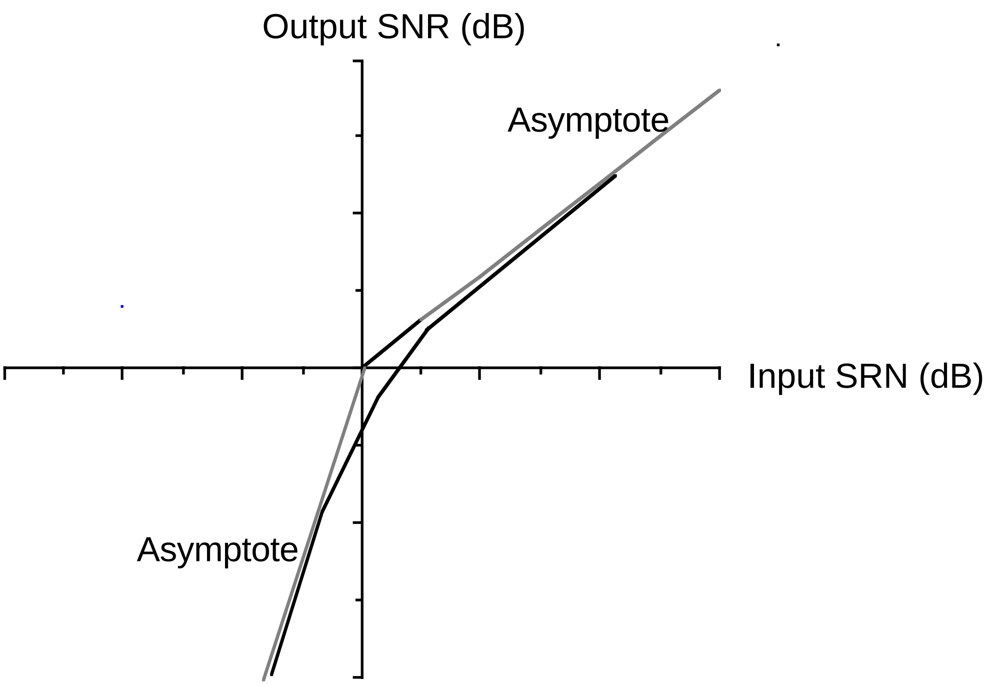

Figure 29.4 Asymptotic signal-to-noise characteristic for envelope detector.

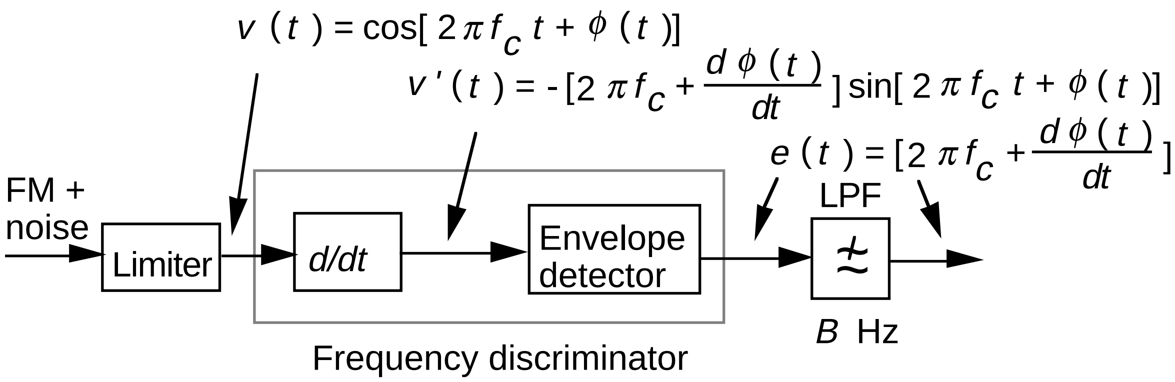

Figure 29.5 Frequency demodulation using a frequency discriminator.

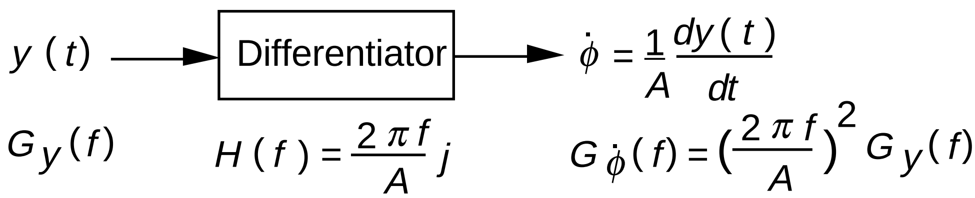

Figure 29.6 Transfer function of a differentiator in an FM receiver.

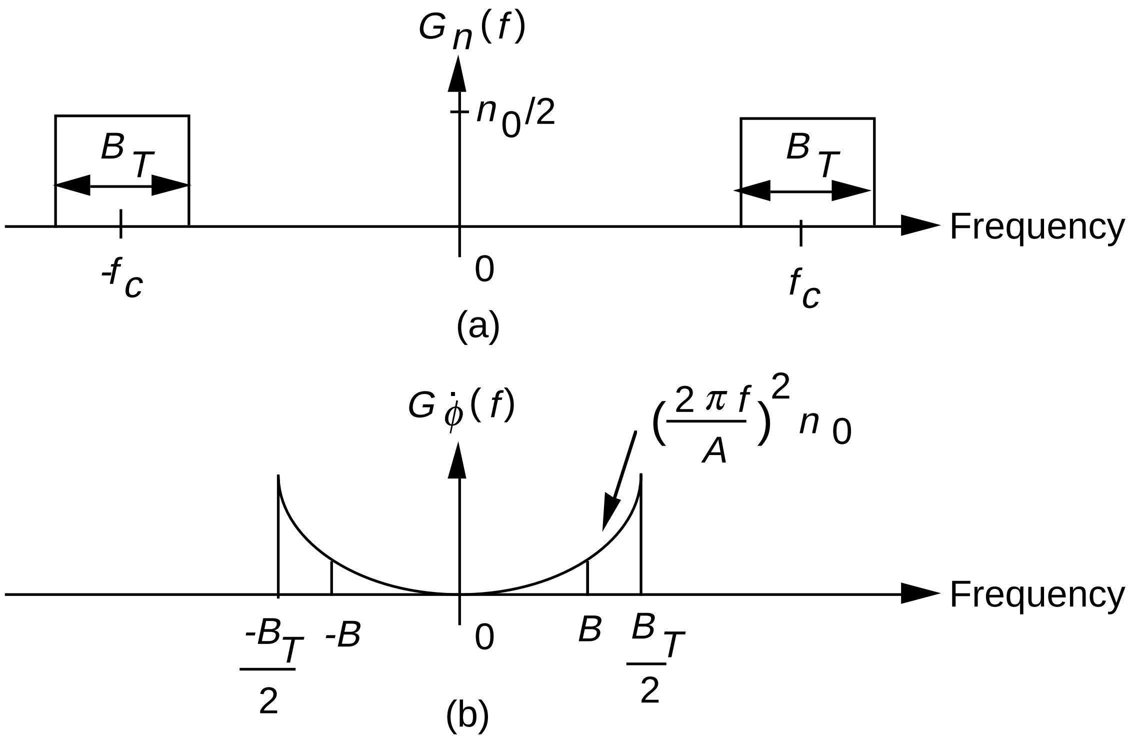

Figure 29.7 Narrowband noise power spectral density.

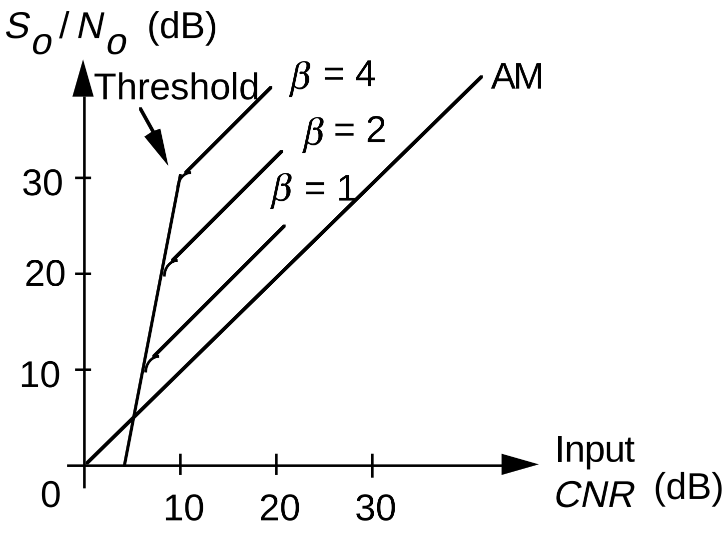

Figure 29.8 Signal-to-noise characteristic for frequency discriminator.