The *z*-transform

Introduction

1. A primary mathematical tool for the analysis and synthesis of digital filters (a special class of linear discrete-time systems).

2. Use to describe a discrete-time sequence.

Definition

Given a discrete-time sequence $\{h(n)\}$, the z-transform of the sequence is defined as

$$H(z) = \sum_{n=-\infty}^{\infty} h(n)\, z^{-n} \qquad (1)$$

where z is a complex-valued variable. $H(z)$ is said to exist for all values of z for which the sum is finite.

In the study of digital filters, we will restrict our attentation to the class of linear time-invariant digital filters, i.e., linear time-invariant systems. The representations of a linear time-invariant system in the time-domain and the z-domain are shown in Fig. 1(a) and (b), respectively.

Fig. 1

Example Let $h(n) = \begin{cases} a^n & \text{for } n \geq 0 \\ 0 & \text{for } n < 0. \end{cases}$

The z-transform $H(z) = \sum_{n=-\infty}^{\infty} a^n z^{-n} = \sum_{n=-\infty}^{\infty} (a z^{-1})^n.$

Given a complex number c, where $|c| < 1$, then the infinite geometric sum

$$\sum_{n=0}^{\infty} c^n = 1/(1 - c) \qquad (2)$$

and $H(z) = (1 + a z^{-1})^{-1}$ for $|a z^{-1}| < 1.$

Properties of the *z*-transform

1. Linearity

The z-transform is a linear operation. Superposition theorem is applied to the definition of the z-transform. The z-transform of a sum of two sequences, $a\{h_1(n)\}$ and $b\{h_2(n)\}$, is the sum of transforms of the individual sequences where a and b are constants. Thus we have

$$\{h(n)\} = \{a\,h_1(n) + a\,h_2(n)\} = \{a\,h_1(n)\} + \{b\,h_2(n)\}$$

and

$$H(z) = \sum_{n=-\infty}^{\infty} h(n)\,z^{-n}$$

$$H(z) = \sum_{n=-\infty}^{\infty} a\,h_1(n)\,z^{-n} + \sum_{n=-\infty}^{\infty} b\,h_2(n)\,z^{-n}$$

$$H(z) = a\,H_1(z) + b\,H_2(z) \qquad (3)$$

2. Shifting

Let $\{h_d(n)\} = \{h(n - d)\}$ be a delayed sequence of the sequence $h(n)$ whose z-transform is given by equation (1). The z-transform of the delayed sequence $\{h_d(n)\}$ is

$$H_d(z) = \sum_{n=-\infty}^{\infty} h_d(n)\,z^{-n}$$

$$H_d(z) = \sum_{n=-\infty}^{\infty} h(n - d)\,z^{-n}$$

Let $m = n - d$, we have

$$H_d(z) = \sum_{m=-\infty}^{\infty} h(m)\,z^{-(m+d)}$$

$$H_d(z) = z^{-d} \sum_{m=-\infty}^{\infty} h(m)\,z^{-m}$$

$$H_d(z) = z^{-d}\,H(z) \qquad (4)$$

3. Convolution

Convolution operation is useful in the analysis of linear system. For the linear time-invariant system shown in Figure 1(a),

$$y(n) = \sum_{k=-\infty}^{\infty} h(k)x(n - k) \qquad (5)$$

$\{y(n)\}$ -- the output sequence
$\{h(k)\}$ -- the unit-sample response sequence
$\{x(n)\}$ -- the input sequence
$\{x(n - k)\}$ -- the delayed sequence of $\{x(n)\}$

Since the z-transform of the sequence $\{y(n)\}$ is

$$\{y(n)\} \;\texttt{<--->}\; Y(z) \tag{6}$$

where

$$Y(z) = \sum_{n=-\infty}^{\infty} y(n)\, z^{-n}$$
$$Y(z) = \sum_{n=-\infty}^{\infty} \Big[\sum_{k=-\infty}^{\infty} h(k)x(n-k) \Big]\, z^{-n}$$
$$Y(z) = \sum_{n=-\infty}^{\infty} \Big[\sum_{k=-\infty}^{\infty} h(k)x(n-k)\Big]\, z^{-n}z^{-k}z^{k}$$
$$Y(z) = \sum_{n=-\infty}^{\infty} x(n-k)\, z^{-(n-k)} \sum_{k=-\infty}^{\infty} h(k)z^{-k}$$

$$Y(z) = X(z)H(z) \tag{7}$$

Therefore,

$$\{y(n)\} = \{h(n)\}*\{x(n)\} \;\texttt{<--->}\; Y(z) = X(z)H(z) \tag{8}$$

4. Scaling

If $\{h(n)\}$ <---> $H(z)$, the z-transform of the sequence $\{a_n\, h(n)\}$ is

$$\sum_{n=-\infty}^{\infty} a_n\, h(n)\, z^{-n} \tag{9}$$

where a is a constant. Let $y^{-1} = a\, z^{-1}$, then equation (9) becomes

$$\sum_{n=-\infty}^{\infty} h(n)y^{-n} = H(y)$$
$$\sum_{n=-\infty}^{\infty} h(n)y^{-n} = H(z/a)$$

Therefore,

$$\{a_n\, h(n)\} \;\text{<--->}\; H(z/a) \qquad (10)$$

5. Multiplication by *n*

If $\{h(n)\}$ <---> $H(z)$, the z-transform of the sequence $\{n\, h(n)\}$ is

$$\sum_{n=-\infty}^{\infty} n\, h(n) z^{-n} = z \sum_{n=-\infty}^{\infty} h(n) z^{-n-1}$$

Since

$$H(z) = \sum_{n=-\infty}^{\infty} h(n) z^{-n}$$

$$\frac{d}{dz} \mathrm{H(z)} = - \sum_{n=-\infty}^{\infty} n\, h(n) z^{-n-1}$$

Therefore,

$$\{n\, h(n)\} \;\text{<--->}\; -z \frac{d}{dz} H(z) \qquad (11)$$

In general,

$$\{n^k\, h(n)\} \;\text{<--->}\; [-z \frac{d}{dz}]^k H(z) \qquad (12)$$

Summary

Definition	$H(z) = \sum_{n=-\infty}^{\infty} h(n) z^{-n}$		
Linearity	$\{h(n)\} = \{a\, h_1(n) + a\, h_2(n)\}$	<--->	$H(z) = a\, H_1(z) + b\, H_2(z)$
Shifting	$\{h_d(n)\} = \{h(n - d)\}$	<--->	$H_d(z) = z^{-d} H(z)$
Convolution	$\{y(n)\} = \{h(n)\} * \{x(n)\}$	<--->	$Y(z) = X(z) H(z)$
Scaling	$\{a_n\, h(n)\}$	<--->	$H(z/a)$
Multiplication	$\{n^k\, h(n)\}$	<--->	$[-z \frac{d}{dz}]^k H(z)$

The Region of Convergence (ROC)

In the definition of the z-transform of a sequence $\{h(n)\}$,

$$H(z) = \sum_{n=-\infty}^{\infty} h(n)\, z^{-n} \,,$$

the concept of a *z*-transform is only useful for values of *z* for which $H(z)$ is finite. The region, where the values of *z* lie, is called the Region of Convergence (ROC). The ROC is defined as a set of z values for which

$$\sum_{n=-\infty}^{\infty} |\, h(n)\, z^{-n} \,| < \infty . \qquad (13)$$

The ROC is important to determine the stability of a linear discrete-time system and the uniqueness of the inverse transform. The inverse of $H(z)$ is not unique unless we specify the ROC.

Example

Consider the two sequences $\{x(n)\}=\{..., 0, \alpha^{n=0}, \alpha^1, \alpha^2, \alpha^3, ...\}$ and $\{y(n)\}=\{..., -\alpha^{-3}, -\alpha^{-2}, -\alpha^{-1}, 0, 0, ...\}$, the z-transforms are

$$X(z) = \sum_{n=0}^{\infty} \alpha^n z^{-n} = (1 - a\, z^{-1})\text{-1},\ |z| > |\alpha|$$

$$Y(z) = \sum_{n=-\infty}^{-1} \alpha^n z^{-n} = (1 - a\, z^{-1})\text{-1},\ |z| < |\alpha| .$$

Thus, distint sequences give rise to the same *z*-transforms.

Fig. 2

The Inverse z-transform

Given $H(z)$ and the ROC, there are several methods to determine the discrete-time sequence $\{h(n)\}$. From the definition of the *z*-transform, $h(n)$ is the coefficient of z^{-n}. If $H(z)$ is expressed as a polynomial in powers of *z*, determine $\{h(n)\}$ is trivial.

Example $H(z) = a + b\, z^{-1}$

$h(n = 0) = a$
$h(n = 1) = b$
$\{h(n)\} = \{a, b\}$

Direct Division Method

If $H(z)$ is given in the form of a rational function of z, a power series in z or z^{-1} can be obtained by long division process.

Advantage -- unless $H(z)$ is relatively simple, the method is easy to apply.

Disadvantage -- it is not possible to obtain a closed form expression.

Example

$H(z) = (z^2 + z)/(z^2 + 0.5z + 1)$
$H(z) = 1 + 0.5z^{-1} - 0.75z^{-2} + \ldots$
$\{h(n)\} = \{1, 0.5, -0.75, \ldots\}$

Partial Fraction Expansion Method

If $H(z)$ is given in the form of a rational function of z, a more useful method for finding the sequence $\{h(n)\}$ is to perform a partial fraction expansion of $H(z)$.

Example 1

$H(z) = (2z^4 + 4z^3 - 14.5z^2 - 44.5z - 33.5) / (z^3 + 1.5z^2 - 8.5z - 15)$, $2.6 < |z| < 2.9$.

$H(z) = 2z + 1 + [(z^2 - 6z - 18.5) / (z^3 + 1.5z^2 - 8.5z - 15)]$

Consider $H_1(z) = (z^2 - 6z - 18.5) / (z^3 + 1.5z^2 - 8.5z - 15)$
$= (z^2 - 6z - 18.5) / [(z + 2)(z + 2.5)(z - 3)]$

$= a + [b\, z/(z + 2)] + [c\, z/(z + 2.5)] + [d\, z/(z - 3)]$

where $a = 1.233$, $b = -0.5$, $c = -0.4$, and $d = -0.333$. Thus

$H_1(z) = 1.233 - [0.5\, z/ (z + 2)] - [0.4z / (z + 2.5)] - [0.333z / (z-3)]$

Therefore,

$H(z) = 2z + 2.233 - [0.5z / (z + 2)] - [0.4z / (z + 2.5)] - [0.333z / (z - 3)]$
$\{h(n)\} = \{2, 2.233, \ldots\}$

Example 2

$H(z) = a(\sin w)z^{-1}/[1 - 2a(\cos w)z^{-1} + a^2z^{-2}]$
$H(z)= [(1 - a\, e^{jw}\, z^{-1})^{-1} - (1 - a\, e^{jw}\, z^{-1})-1] / 2j$

From equation (2), $\sum_{n=0}^{\infty} c^n = (1 - c)^{-n}$, then

$$H(z) = \frac{1}{2j}\left[\sum_{n=0}^{\infty} (a\, e^{jw}\, z^{-1})^n - \sum_{n=0}^{\infty} (a\, e^{-jw}\, z^{-1})^n\right]$$

$$H(z) = \sum_{n=0}^{\infty} a^n (\sin nw) z^{-n}$$

$$\{h(n)\} = \{0, a(\sin w), a^2(\sin 2w), \ldots, a^n(\sin nw), \ldots\}$$

The Inversion Integral

The inverse of H(z) is based on the Cauchy integral formula. Consider the z-transform

$$H(z) = \sum_{n=-\infty}^{\infty} h(n)\, z^{-n} \qquad (14)$$

in the region of the z-plane for which the series converges. From complex-variable theory, it can be shown that

$$H(z) = \frac{1}{2\pi j} \oint_c H(z) z^{n-1}\, dz \qquad (15)$$

where the integration is taken in a counterclockwise direction along a circular contour c that lies in the region of convergence and encloses the origin. For rational z-transforms contour integrals such as equation (15) are often found by the application of the Residue Theorem.

Residue Theorem : If $H(z)z^{n-1}$ has N poles then

$$h(n) = \sum_{i=1}^{N} \text{Res}[H(z)z^{n-1} \text{ at pole } p_i \text{ inside } c] \qquad (16)$$

If $H(z)z^{n-1}$ is a rational function of z and has a pole $z = z_0$ of magnitude (order) m, then the residue at the pole is

$$\text{Res}\{H(z)z^{n-1} \text{ at } z = z_0\} = \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} (z - z_0)^m . H(z)z^{n-1} | \quad (17)$$

Example

$H(z) = z + a + z^{-1}$, for n = -1

$H(z)z^{-2} = z^{-1} + a\, z^{-2} + z^{-3}$

$H(z)z^{-2} = (z^2 + a\, z + 1) / z^3$

$(z - 0)^3 . H(z)z^{-2} = z^2 + a\, z + 1$ a 3rd-order pole at $z = 0$!

$$\frac{d^2}{2!dz^2} (z - 0)^3 . H(z)z^{-2} = 2 / 2!$$

$$\frac{d^2}{2!dz^2} (z - 0)^3 . H(z)z^{-2} = 1$$

$$\frac{d^2}{2!dz^2} (z - 0)^3 . H(z)z^{-2} = h(-1)$$

Example 2

$H(z) = 1 / (1 - a\, z^{-1})$, ROC : $|z| > a$. For n = -1, find $h(-1)$.

$H(z)z^{-2} = 1/[z(z - a)]$ two 1st-order poles at $z = 0$ and $z = a$!

$$\text{Res}\{H(z)z^{n-1} \text{ at } z = 0\} = \frac{d^0}{dz^0} (z - 0)H(z)z^{-2} |_{z = 0}$$

$$\text{Res}\{H(z)z^{n-1} \text{ at } z = 0\} = \frac{d^0}{dz^0} 1/(z - a) |_{z = 0}$$

$$\text{Res}\{H(z)z^{n-1} \text{ at } z = 0\} = -1 / a$$

$$\text{Res}\{H(z)z^{n-1} \text{ at } z = a\} = \frac{d^0}{dz^0} (z - 0)H(z)z^{-2} |_{z = a}$$

$$\text{Res}\{H(z)z^{n-1} \text{ at } z = a\} = \frac{d^0}{dz^0} 1/z |_{z = a}$$

$$\text{Res}\{H(z)z^{n-1} \text{ at } z = a\} = 1/a$$

Therefore, by equation (16),

$h(-1) = -1/a + 1/\text{i}$

$h(-1) = 0$

z-transform of Symmetric Sequences

1. Find the z-transform of an even sequence $\{h(n)\}$ that has its point ofsymmetry falls on a sampling point as shown in Fig. 3.

Fig. 3

$$H_E(z) = \sum_{n=-m}^{m} h(n)\, z^{-n}$$

$$H_E(z) = \sum_{n=-m}^{-1} h(n)\, z^{-n} + h(0) + \sum_{n=1}^{m} h(n)\, z^{-n}$$

$$H_E(z) = \mathrm{h}(0) + \sum_{n=1}^{m} h(n)\,(z^{n} + z^{-n}) \qquad (18)$$

If z is replaced by 1/z, there is no change in eqn. (18). Thus

$$H_E(z) = H_E(1/z) \qquad (19)$$

A causal even-symmetric sequence $\{h_S(n)\} = \{h(n - m)\}$ can then be obtained from $\{h(n)\}$ with a delay of m samples. Using the shifting property, the z-transform of $\{h_S(n)\}$ is

$$H_S(z) = z^{-m} H_E(z) \qquad (20)$$

2. Find the z-transform of an odd sequence $\{h(n)\}$ that has its point ofsymmetry falls on a sampling point as shown in Fig. 4.

Fig. 4

$$H_O(z) = \sum_{n=-m}^{m} h(n)\, z^{-n}$$

$$H_O(z) = \sum_{n=-m}^{-1} h(n)\, z^{-n} + h(0) + \sum_{n=1}^{m} h(n)\, z^{-n}$$

$$H_O(z) = \sum_{n=1}^{m} h(n)\,(z^{-n} - z^{n}) \qquad (18)$$

If z is replaced by 1/z, we get

$$H_O(1/z) = \sum_{n=1}^{m} h(n)\,(z^{n} - z^{-n})$$

Therefore,

$$H_O(1/z) = -H_O(z). \qquad (22)$$

Relationship between the *z*-transform and the DTFT

The complex variable z can be expressed in the complex-vector form given by

$$z = r\, e^{j\theta}. \tag{23}$$

Eqn. (1) can be expressed as

$$H(r\, e^{j\theta}) = \sum_{n=-\infty}^{\infty} h(n)\, r^{-n}\, e^{j\theta n}. \tag{24}$$

When $r = 1$, eqn. (24) becomes

$$H(e^{j\theta}) = \sum_{n=-\infty}^{\infty} h(n)\, e^{j\theta n}. \tag{25}$$

The connection between $H(e^{j\theta})$ is that the DTFT $H(e^{j\theta})$ is $H(z)$ restricted to the unit circle. At a particular value of $\theta = \theta_0$,

$$H(e^{j\theta_0}) = H(z)\,\big|_{e^{j\theta_0}} \tag{26}$$

which corresponds to the intersecting point between the unit-circle and the line passing through the origin with an angle of θ_0 w.r.t. the real axis in the *z* plane.

Example

$$H(z) = 1/(1 - a\, z^{-1})$$

$$H(z) = z/(z - a)$$

$$H(e^{j\theta_0}) = e^{j\theta_0} / (e^{j\theta_0} - a)$$

$$H(e^{j\theta_0}) = (\cos\theta_0 + j\sin\theta_0)/(\cos\theta_0 - a + j\sin\theta_0).$$

Realisation of a Digital Filter from the System Function *H*(*z*)

Digital filters (a class of linear system) can be realised in an infinite number of ways. In general, we can describe the output of a finite-order linear time-invariant (LTI) system as a linear combination of the inputs and outputs in the following form :

$$y(n) = \sum_{l=1}^{M} a_l \; y(n - l) + \sum_{l=-N_f}^{N_p} b_l \; x(n - l). \tag{27}$$

$y(n)$ -- current output at time n
$y(n - l)$ -- past output
a_l -- feedback coefficient
$(n - l)$ -- past input
b_l -- feedforward coefficient

The z-transform of equation (27) is

$$Y(z) = \sum_{n=-\infty}^{\infty} y(n)\, z^{-n}$$

$$Y(z) = \sum_{n=-\infty}^{\infty} \left[\sum_{l=1}^{M} a_l \; y(n - l)\, z^{-n} + \sum_{l=-N_f}^{N_p} b_l \; x(n - l)\, z^{-n} \right]$$

$$Y(z) = \sum_{l=1}^{M} a_l\, z^{-l}\, Y(z) + \sum_{l=-N_f}^{N_p} b_l\, z^{-l}\, X(z) \tag{28}$$

$$Y(z) = X(z) \sum_{l=-N_f}^{N_p} b_l\, z^{-l} / [1- \sum_{l=1}^{M} a_l\, z^{-l}]$$

$$H(z) = Y(z)/X(z)$$

$$H(z) = \sum_{l=-N_f}^{N_p} b_l\, z^{-l} / [1- \sum_{l=1}^{M} a_l\, z^{-l}] \tag{29}$$

Example

$$y(n) = a_1 y(n - 1) + a_2 y(n - 2) + b_0(n)x(n) + b_1 x(n - 1)$$

$$H(z) = (b_0 + b_1 z^{-1})/(1 - a_1 z^{-1} - a_2 z^{-2})$$

A digital filter can be implemented from the general difference equation (equation (28)), as shown in Fig. 5.

Fig. 5

There are two common ways of combining system function. When the system functions are connected in series, the overall system function $H_c(z)$ is simply the product of each system function, i.e.,

$$H_c(z) = \prod_{n=1}^{N} H_n(z) . \tag{30}$$

When the system functions are connected in parallel, the overall system function $H_p(z)$ is simply the sum of each system functions; i.e.,

$$H_p(z) = \sum_{n=1}^{N} H_n(z). \tag{31}$$

Poles and Zeros Representation of $H(z)$ in the z-plane

The z-transform $H(z)$ is a function of the complex-valued variable z. In this section, we consider $H(z)$ in terms of its singularities in the z-plane. Consider the system function given by

$$H(z) = \sum_{l=-N_f}^{N_p} b_l z^{-l} /[1 - \sum_{l=1}^{M} a_l z^{-l}] \tag{32}$$

which can be factorised into the following product-of-terms form :

$$H(z) = A z^{N_f} \sum_{l=1}^{N_p+N_f} (1 - c_l z^{-l}) / \sum_{l=1}^{M} (1 - d_l z^{-l}) \tag{33}$$

where

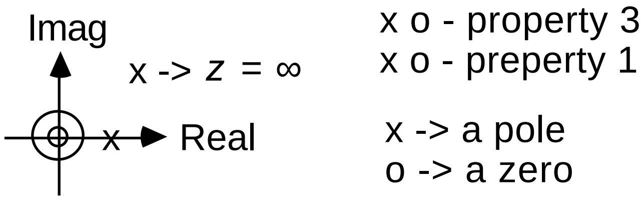
A -- a real-valued constant
c_l -- the location of the l-th zero
d_l -- the location of the l-th pole.

Equation (33) has the following properties :

1. The term z^{N_f} generates N_f poles at $z = \infty$ and N_f zeros at $z = 0$.
2. If $N_f > 0$, $H(z)$ is noncausal.
3. Each denominator term generates a pole at $z = d_l$ and a zero at $z = 0$.
4. Each numerator term $(1 - c_l z^{-l})$ generates a zero at $z = c_l$ and a pole at $z = 0$.

5. The total number of poles is equal to the total number of zeros.

<u>Example</u> $H(z) = z/(z - a)$



We have already seen that the z-transform is only useful for values of z for which $H(z)$ is finite. Since $H(z)$ becomes infinite at the location of the poles, the location of the poles will determine the boundaries of the Region of Convergence (ROC). Consider the term $(1 - d_l z^{-l})^{-1}$, it can be expressed in an infinite geometric sum provided |d z-1| < 1 (see equation (2)). For $|d_l z^{-l}| < 1$,

$$(1 - d_l z^{-l})^{-1} = \sum_{n=0}^{\infty} d^n z^{-n} \tag{34}$$

$$(1 - d_l z^{-l})^{-1} = \sum_{n=0}^{\infty} h(n) z^{-n}$$

For a stable system,

$$\sum_{n=0}^{\infty} |h(n)| < \infty. \tag{35}$$

For $|d_l| < 1$, $\sum_{n=0}^{\infty} |d_l|^n = (1 - d_l)^{-1}$ and the system becomes a stable system.

It can be seen that each pole of a stable system must be lie within the unit circle, i.e.,

<u>Fig. 6</u>

<u>Summary</u>

1. $N_f \leq 0$ gives a causal system.
2. $N_f > 0$ gives a non-causal system.
3. $d_l > 1$ gives a non-stable system.
4. $d_l < 1$ gives a stable system.

<u>Geometrical Evaluation of $H(z)$ and its Application</u>

In general, the evaluation of the z-transform $H(z = r\, e^{j\theta})$ at ANY POINT in the z-plane can be done in a geometrical way. Consider equation (33), the magnitude is given by

$$| H(z = z_0) | = |A| \prod_{l=1}^{K_z} Z_l \, / \prod_{l=1}^{K_p} P_l \qquad (36)$$

and the phase is

$$\text{Arg}\,[H(z{=}z0)] = \sum_{l=1}^{K_z} Z_l - \sum_{l=1}^{K_p} P_l \qquad (37)$$

where K_z is the number of zeros, K_p is the number of poles, Z_l is the distance from the l-th zero to the point $z = z_0$ and P_l is the distance from the l-th pole to the point $z = z_0$.

Example

Fig. 7

Application

The approximate shape of the magnitude response of $H(e^{j\theta})$ can be geometrically evaluated from the pole/zero pattern. If the pole/zero is close to the unit-circle in the z-plane, the distance from it can change radically for small changes in h. This is shown in Fig. 8.

Fig. 8

From Fig. 8, the following points are noted.

1. Poles/zeros at the origin do not contribute to the magnitude response because the distance from the origin to the unit circle is 1. However, poles/zeros at the origin do contribute to the phase response.

2. If a zero lies inside the unit circle, Z_l increases by π radians as θ increases from 0 to π.

3. If a pole lies inside the unit circle, P_l decreases by π radians as θ increases

from 0 to π. The net increase in phase as θ increases from 0 to π is $(K_Z - K_p)\pi$.

4. A phase jump of π results as the frequency passes through each zero lying on the unit circle.

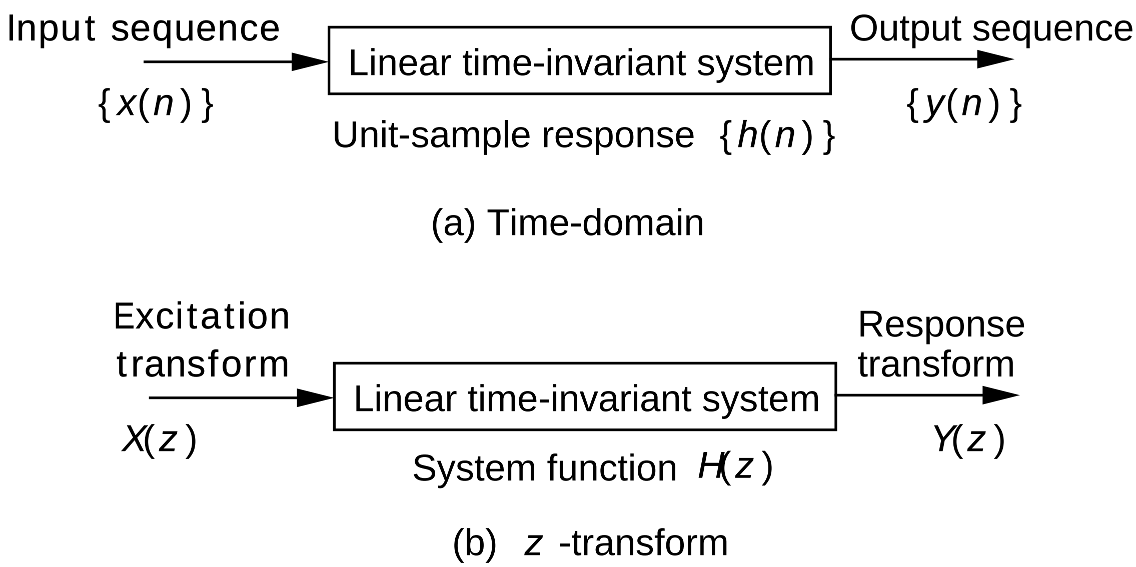


Figure 1

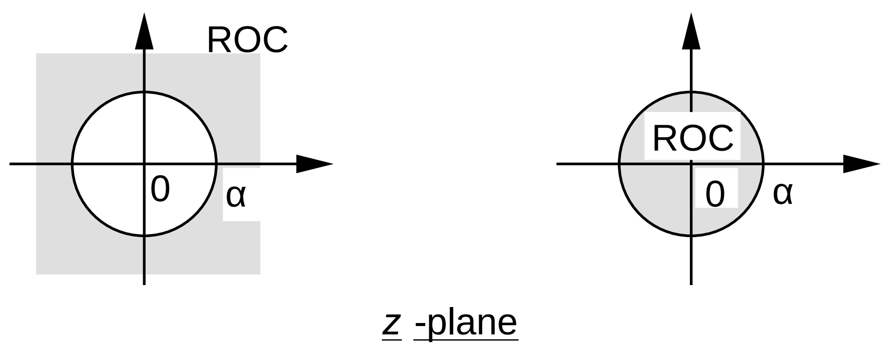


Figure 2

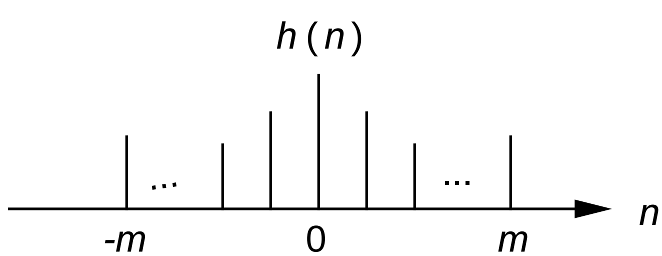


Figure 3

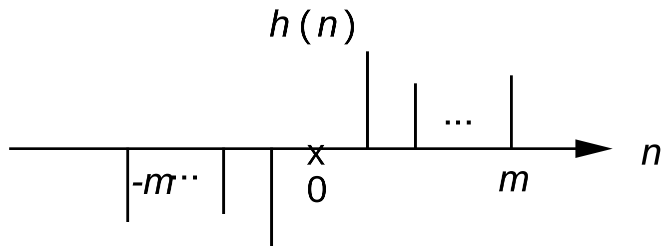


Figure 4

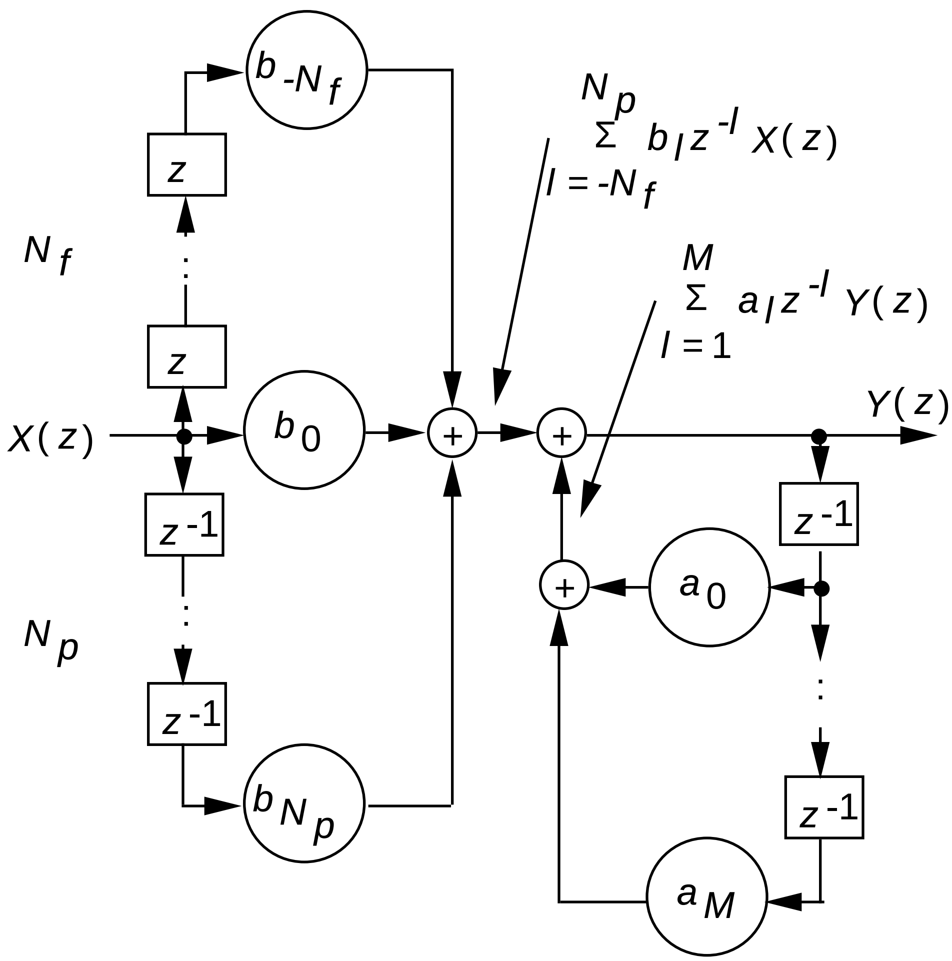


Figure 5

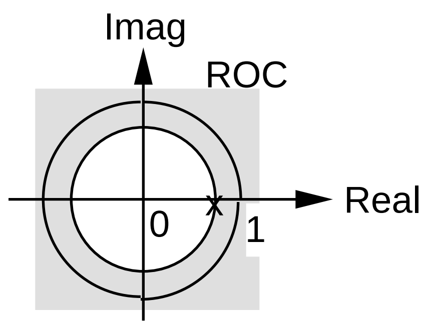


Figure 6

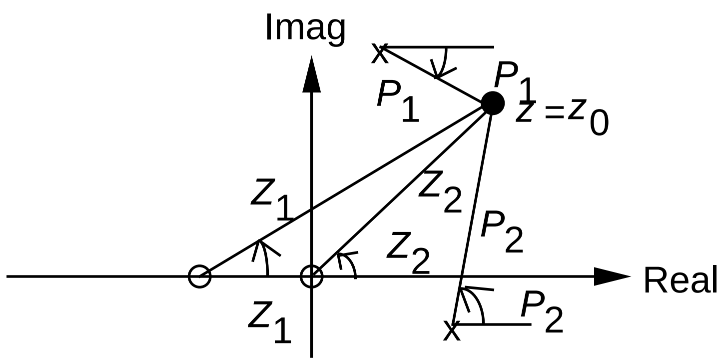


Figure 7

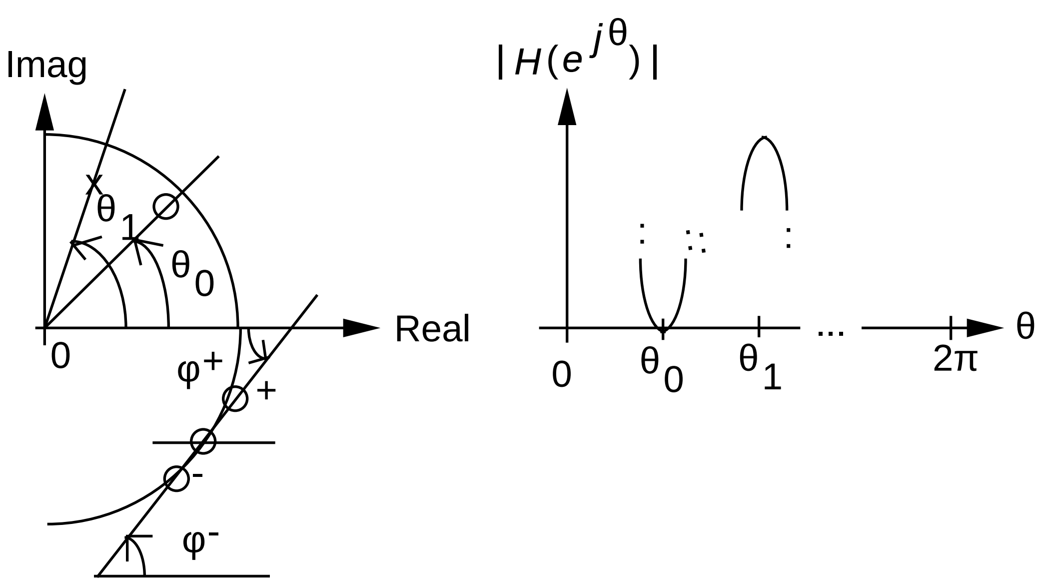


Figure 8