Random Processes

Introduction

To analysis and design a communication system whose input is a collection of waveforms, a probabilistic model (similar to the concept of random variables), can be used to describe the waveforms (functions of time). In this way, we can predict the output of a system responding to a random process at its input.

Definitions

A random process can be considered to be a signal which varies in a random manner with time. The behaviour of the signal cannot be predicted with certainty.

The properties of random processes are analysed mathematically by assuming that the actual variation of the process with time, over some time intervals 0 to T seconds. The concept of a random process is based on enlarging the random variable concept to include time. Since a random variable X is a function of the possible outcome e of an experiment, we assign, according to some rules, a time function $x(t, e)$ to every outcome e. The family of all such functions, denoted $X(t, e)$, is called a random process.

The short-form notation $x(t)$ is used to represent a specific waveform of a random process $X(t, e)$.

Example

Figure 5

From Figure 5, we can see that a real valued time function $X(t, e)$ is assigned to the outcome e.

$$x_1(t) = X(t, e_1)$$

$$x_2(t) = X(t, e_2)$$

$$x_3(t) = X(t, e_3).$$

We have thus created a set of functions. This set (a collection of functions) is a random process.

The totality of all functions is called an ensemble when t and e are variables.

A time function, selected from the ensemble, is called a sample function. Thus, a specific outcome e_i gives a single sample function $X(t, e_i)$.

At time $t = t_0$, $X(t_0, e)$ represents a random variable whose value depends on the outcome e (i.e., when t is fixed and e is a variable).

Finally, $X(t_0, e_i)$ represents a number when t and e are fixed at t_0 and e_i, respectively.

For example, $X(\tau, e)$ is a random variable (a vertical slice through the ensemble at time $t = \tau$, as illustrated in Figure 5) whose possible values are $x_1(\tau)$, $x_2(\tau)$ and $x_3(\tau)$.

The statistical properties of $X(\tau, e)$ describe the statistical properties of the random process at time $t = \tau$. The expected value of $X(\tau, e)$ is called the ensemble average of the random process at time τ.

The random process can thus be considered as a collection of an infinite number of random variables $\{X(t_i, e)\}$, where t_i takes on all values in the range 0 to T.

Summary

1. $X(t, e)$ in short, $X(t)$ - a random process (a collection of time functions).

2. $X(t, e_i)$ in short, $x_i(t)$ - the i-th time function.

3. $X(t_0, e)$ in short, $X(t_0)$ or X_0 - a random variable at $t = t_0$.

4. $X(t_0, e_i)$ in short, x - a value of the random variable $X(t_0, e)$.

A random process becomes a random variable when time t is fixed at some value t_0.

Descriptions of a Random Process

1. Analytical

   $X(t) = g(Y_1, Y_2, \ldots; t)$.

   Random process $X(t)$ is a function of random variables.

2. Statistical averages (ensemble averages)

Random process can be described in terms of the mean $\mu_X(t)$ and the auto-correlation function $R_{XX}(t_1, t_2)$, defined as

$$\mu_X(t) = E[X(t)] \tag{155}$$

$$R_{XX}(t_1, t_2) = E[X(t_1)\, X(t_2)]. \tag{156}$$

Stationarity

To define stationarity, we must first define distribution and density functions as they apply to a random process $X(t)$.

For a particular time t_1, the distribution function associated with the random variable $X_1 = X(t_1) = X(t_1, e)$ is

$$F_X(x_1; t_1) = P\{X(t_1) \leq x_1\} \tag{157}$$

and the density function is

$$f_X(x_1; t_1) = dF_X(x_1; t_1) / dx_1. \tag{158}$$

1. A random process is called stationary to order one if

$$f_X(x_1; t_1) = f_X(x_1; t_1+\Delta). \tag{159}$$

$f_X(x_1; t_1)$ is independent of t_1 and the process mean value $E[X(t)]$ is

$$\mu_X(t) = E[X(t)] = \text{constant}. \tag{160}$$

2. A random process is called stationary to order two if

$$f_X(x_1, x_2 ; t_1, t_2) = f_X(x_1, x_2 ; t_1+\Delta, t_2+\Delta). \tag{161}$$

Equation (161) is, in fact, a function of time differences $t_2 - t_1$ and not an absolute time.

The auto-correlation $R_{XX}(t_1, t_2)$ is

$$R_{XX}(t_1, t_2) = E[X(t_1)X(t_2)]. \tag{162}$$

A consequence of (161) is that $R_{XX}(t_1, t_2)$ is a function only of time differences and not an absolute time; i.e., if

$$\tau = t_2 - t_1 \tag{163}$$

then (162) becomes

$$R_{XX}(t_1, t_1+\tau) = E[X(t_1)X(t_1+\tau)]$$

$$R_{XX}(t_1, t_1+\tau) = R_{XX}(\tau). \tag{164}$$

The process mean value $E[X(t)]$ is

$$\mu_X(t) = E[X(t)] = \text{constant}. \tag{165}$$

Cases 1 and 2 are called wide-sense stationary process.

In general, second-order wide-sense stationary process satisfies :

$$E[X(t)] = \text{constant} \tag{166}$$

$$R_{XX}(\tau) = E[X(t)X(t+\tau)]. \tag{167}$$

3. A random process is called stationary to order N if

$$f_X(x_1, \ldots, x_N ; t_1, \ldots, t_N) = f_X(x_1, \ldots, x_N; t_1+\Delta, \ldots, t_N+\Delta) \tag{168}$$

for all $t_1, \ldots, t_N$ and Δ. Stationarity to all orders $N = 1, 2, \ldots,$ is called strict-sense stationary process.

Time Averages and Ergodicity

The time average of a quantity is defined as

$$<.> = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} <.>dt. \tag{169}$$

It is now possible to define the auto-correlation function using time averaging; i.e.,

$$<R_{XX}(\tau)> = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} X(t)X(t+\tau)dt. \tag{170}$$

Similarly, the time-averaged mean is

$$<\mu_X(t)> = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} X(t)dt. \tag{171}$$

It can be seen that the values of $<\mu_X(t)>$ and $<R_{XX}(\tau)>$ depend on which member of the time functions is used in the time averaging process. $<\mu_X(t)>$ and $<R_{XX}(\tau)>$are themselves random variables. In contrast, $\mu_X(t)$ is a constant and $R_{XX}(\tau)$ is a function of τ in the ensemble averaging approach (equations (166) and (167)).

In general, time averages are not equal to the ensemble averages (statistical averages).

A random process is said to be ergodic if the time averages equal the ensemble averages; i.e.,

$$E[<\mu_X(t)>] = <\mu_X(t)> = E[X(t)] = \text{constant} \tag{172}$$

and

$$E[\ <R_{XX}(\tau)>\ ] = <R_{XX}(\tau)> = R_{XX}(\tau). \tag{173}$$

Ergodicity is a very restrictive form of stationarity. An ergodic process must be stationary but a stationary process need not be ergodic.

If $X(t)$ is at least wide-sense stationary, $R_{XX}(\tau) = E[X(t)X(t+\tau)]$ is true for wide-sense stationary processes.

For such stationary processes the auto-correlation function exhibits the following properties :

1. $R_{XX}(0) \geq |R_{XX}(\tau)|$. (174)

2. $R_{XX}(\tau) = R_{XX}(-\tau)$. (175)

3. $R_{XX}(0) = E[X^2(t)]$ (176)

$$R_{XX}(0) = \int_{-\infty}^{\infty} G_{XX}(f)df. \tag{177}$$

where $G_{XX}(f)$ is the power spectral density of $X(t)$. The first property shows that $R_{XX}(\tau)$ is upper bounded by its value at $\tau = 0$. The second property shows that $R_{XX}(\tau)$ is real and even. The third property shows that $R_{XX}(0)$ is equal to the mean-squared value called the power in the process.

Further properties may also be stated :

1. If $E[X(t)] \neq 0$ and $X(t)$ has no periodic components then

$$\lim_{|\tau| \to \infty} R_{XX}(\tau) = (E[X(t)])^2.$$

2. If $X(t)$ has a periodic component, then $R_{XX}(\tau)$ will have a periodic component with the same period.

3. If $X(t)$ is ergodic, zero-mean, and has no periodic component, then

$$\lim_{|\tau| \to \infty} R_{XX}(\tau) = 0.$$

We shall consider only stationary random processes which are ergodic and real.

Mean $\mu_X(t) = E[X(t)]$

Auto-correlation $R_{XX}(\tau) = E[X(t)X(t+\tau)]$

Cross-correlation $R_{XY}(\tau) = E[X(t)Y(t+\tau)]$

Auto-covariance

The concept of the covariance of two random variables can be extended to random processes. The auto-covariance function is defined by

$$C_{XX}(t, t+\tau) = E[\{X(t) - E[X(t)]\} \{X(t+\tau) - E[X(t+\tau)]\}] \tag{178}$$

$$C_{XX}(t, t+\tau) = E\{X(t)X(t+\tau) - X(t)E[X(t+\tau)] -$$

$$X(t+\tau)E[X(t)] + E[X(t)]E[X(t+\tau)]\}$$

$$C_{XX}(t,\ t+\tau) = E[X(t)X(t+\tau)] - E[X(t)]\ E[X(t+\tau)]$$

$$C_{XX}(t,\ t+\tau) = R_{XX}(t,\ t+\tau) - \ E[X(t)]\ E[X(t+\tau)].$$

For processes that are at least wide-sense stationary, equation (178) becomes

$$C_{XX}(\tau) = R_{XX}(\tau) - \mu_X(t)^2. \tag{179}$$

The variance of a random process is given in general by (178) with $\tau = 0$.

$$C_{XX}(0) = \sigma_X{}^2 = R_{XX}(0) - \mu_X(t)^2. \tag{180}$$

If $C_{XX}(\tau) = 0$, then, from (179),

$$R_{XX}(\tau) = \mu_X(t)^2 \tag{181}$$

$$R_{XX}(\tau) = E[X(t)]\ E[X(t+\tau)].$$

N.B. :- $E[X(t)] = E[X(t+\tau)]$ for stationarity.

Two random variables, given by the random process $X(t)$ at the times t and $t+\tau$ respectively, are uncorrelated when

$$R_{XX}(\tau) = \mu_X(t)^2 \ \underline{\text{or}}\ C_{XX}(\tau) = 0.$$

If $R_{XX}(\tau) = 0$, the random variables are orthogonal.

Cross-covariance

The cross-covariance function for two processes $X(t)$ and $Y(t)$ is defined by

$$C_{XY}(t,\ t+\tau) = E[\{X(t)-E[X(t)]\}\ \{Y(t+\tau)-E[Y(t+\tau)]\}]$$

$$C_{XY}(t,\ t+\tau) = R_{XY}(t,\ t+\tau) - E[X(t)]\ E[Y(t+\tau)].$$

Two random processes $X(t)$ and $Y(t)$ are uncorrelated when

$$R_{XY}(\tau) = E[X(t)]\ E[Y(t+\tau)]$$

<u>or</u>

$$C_{XY}(\tau) = 0.$$

If $R_{XY}(\tau) = 0$, the random processes are <u>orthogonal</u>.

Gaussian Random Processes

The greatest theoretical and practical importance is the Gaussian random process. Consider a continuous random process and define N random variables $X_1 = X(t_1)$, ..., $X_i = X(t_i)$, ..., $X_N = X(t_N)$ corresponding to N time instants t_1, ..., t_i, ..., t_N.

If, for any N and times t_1, ..., t_N, these random variables are jointly Gaussian, the process is called Gaussian and

$$f_X(x_1, .., x_N; t_1, .., t_N) = \frac{\exp\{-\frac{1}{2}[x-\mu_X(t)]^t[C_X]^{-1}[x-\mu_X(t)]\}}{\sqrt{(2\pi)^N \left|[C_X]\right|}} \qquad (183)$$

where matrix

$$[x - \mu_X(t)] = \begin{bmatrix} x_1 - \mu_X(t_1) \\ x_2 - \mu_X(t_2) \\ \vdots \\ x_N - \mu_X(t_N) \end{bmatrix} \qquad (184)$$

and

$$[C_X] = \begin{bmatrix} C_{11} & C_{12} & & C_{1N} \\ C_{21} & C_{22} & & C_{2N} \\ \vdots & & & \vdots \\ C_{N1} & C_{N2} & & C_{NN} \end{bmatrix} \qquad (185)$$

$[.]^t$ denotes the transpose of a matrix,

$[.]^{-1}$ denotes the inverse of a matrix,

$|[.]|$ denotes the determinant of a matrix.

$[C_X]$ is called the covariance matrix of the N random variables, where

$$C_{ij} = E[\{X(t_i)-E[X(t_i)]\}\ \{X(t_j)-E[X(t_j)]\}]$$
$$C_{ij} = E[\{X(t_i)-E[X(t_i)]\}\ \{X(t_j)-\mu_X(t_j)]\}]$$
$$C_{ij} = C_{XX}(t_i,\ t_j). \tag{186}$$

In fact,

$$C_{ij} = \begin{cases} \sigma^2_{X_i} & i = j \\ C_{X_i X_j} & i \neq j \end{cases} \tag{187}$$

From (183) and (186), a Gaussian random process is COMPLETELY defined by its mean and auto-correlation function. We state without proof some of the properties exhibited by N jointly Gaussian random variables $X_1, \ldots, X_N$.

1. If the mean and the auto-correlation function are stationary, then the joint densities of <u>all orders</u> are functions only of time differences and so are also stationary.

2. If a Gaussian random process is stationary in the wide sense, it is also strictly stationary.

The most important Gaussian random process in the engineering field is the first- and second- order stationary Gaussian random processes.

1. For a single Gaussian random process $X(t)$

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2_X}} \exp\left[\frac{-(x-\mu_X)^2}{2\sigma^2_X}\right]. \tag{188}$$

2. For a zero-mean Gaussian random process $X(t)$

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2_X}} \exp\left[\frac{-x^2}{2\sigma^2_X}\right]. \tag{189}$$

Since $R_{XX}(\tau) = E[X(t)\,X(t+\tau)]$

$$R_{XX}(0) = E[X(t)^2]$$

$$R_{XX}(0) = \sigma_X^2 .$$

Therefore,

$$f_X(x) = \frac{1}{\sqrt{2\pi R_{XX}(0)}} \exp\left(\frac{-x^2}{2R_{XX}(0)}\right). \tag{190}$$

3. For a second-order Gaussian random process $X(t)$ of $N = 2$ random variables X_1 and X_2,

$$f_{X_1,X_2}(x_1, x_2; t_1, t_2) = \frac{1}{2\pi\sigma_{X_1}\sigma_{X_2}\sqrt{1-\rho^2}}$$

$$\exp\left\{\frac{-1}{2(1-\rho^2)}\left[\frac{(x_1-\mu_X(t_1))^2}{\sigma_{X_1}^2} - \frac{2\rho(x_1-\mu_X(t_1))(x_2-\mu_X(t_2))}{\sigma_{X_1}\sigma_{X_2}} + \frac{(x_2-\mu_X(t_2))^2}{\sigma_{X_2}^2}\right]\right\} \tag{191}$$

where

$$\mu_X(t_1) = E[X_1] \tag{192}$$

$$\mu_X(t_2) = E[X_2] \tag{193}$$

$$\sigma^2_{X_1} = E\{[X_1 - \mu_X(t_1)]^2\} \tag{194}$$

$$\sigma^2_{X_2} = E\{[X_2 - \mu_X(t_2)]^2\} \tag{195}$$

$$\rho = \frac{E\{[X_1 - \mu_X(t_1)][X_2 - \mu_X(t_2)]\}}{\sigma_{X_1}\sigma_{X_2}}. \tag{196}$$

4. For a second-order zero-mean Gaussian random process $X(t)$ of $N = 2$ random variables X_1 and X_2,

$$f_{X_1,X_2}(x_1, x_2; t_1, t_2) =$$

$$\frac{1}{2\pi\sigma_{X_1}\sigma_{X_2}\sqrt{1-\rho^2}} \exp\left\{\frac{-1}{2(1-\rho^2)}\left[\frac{x_1^2}{\sigma^2_{x_1}} - \frac{2\rho x_1 x_2}{\sigma_{X_1}\sigma_{X_2}} + \frac{x_2^2}{\sigma^2_{x_2}}\right]\right\}. \tag{197}$$

<u>Spectral Representation of Random Processes</u>

So far we have characterised processes by means of auto-correlation, cross-correlation, and covariance functions which involved the time-domain. It is, in fact, possible to describe random processes in the frequency domain.

Recall the Fourier Transform pair for a deterministic signal $x(t)$, where

$$X(w) = \int_{-\infty}^{\infty} x(t)\, e^{-jwt}\, dt \tag{198}$$

and

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(w)\, e^{jwt}\, dw. \tag{199}$$

In attempting to apply (198) to a random process, we have encountered problems. $X(w)$ may not exist for most sample functions (<u>random in nature</u>) of the process.

Thus, a spectral description of a random process utilising an amplitude density spectrum (Fourier Transform) is not feasible because such a spectrum may not exist. If we turn our attention to the description of the <u>power</u> in the random process as a function of frequency, it results that such a function does exist.

Power Density Spectrum (Power Spectral Density)

Let $x_T(t)$ be a portion of a sample function $x(t)$ of a random process $X(t)$ such that

$$x_T(t) = \begin{cases} x(t) & -T < t < T \\ 0 & eslewhere. \end{cases} \tag{200}$$

For finite T,

$$\int_{-T}^{T} |x_T(t)|\, dt < \infty \tag{201}$$

The Fourier transform of $x_T(t)$ is

$$X_T(w) = \int_{-T}^{T} x_T(t)\, e^{-jwt}\, dt = \int_{-T}^{T} x(t) e^{-jwt}\, dt. \tag{202}$$

The average power in $x(t)$ over the interval $(-T, T)$ is

$$P(T) = \frac{1}{2T} \int_{-T}^{T} x^2(t)\, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{|X_T(w)|^2}{2T}\, dw. \tag{203}$$

PARSEVAL'S THEOREM

From (203), $|X_T(w)|^2/2T$ is a power density spectrum. However, (203) does not represent the power in an entire sample function. T must be extended to infinity. (203) only represents the power in one sample function and does not represent the random process. $P(T)$ is actually a random variable with respect to the random process. By taking the expected value in (203), we can obtain an average power P_{XX} for the random process.

$$P_{XX} = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} E[X^2(t)]dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} \lim_{T \to \infty} \frac{E[|X_T(w)|]^2}{2T}\, dw. \tag{204}$$

From (169), $P_{XX} = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} E[X^2(t)]dt$ represents the time average of $E[X^2(t)]$. For a wide-sense stationary random process, $E[X^2(t)]$ is a constant and $P_{XX} = E[X^2(t)]$, a constant. If we define the power spectral density for the random process $X(t)$ by

$$G_{XX}(w) = \lim_{T \to \infty} \frac{E[\left|X_T(w)\right|]^2}{2T} \, . \qquad (205)$$

Then

$$P_{XX} = \frac{1}{2\pi} \int_{-\infty}^{\infty} G_{XX}(w)\, dw \, . \qquad (206)$$

The power spectral density has the following properties :

1. $G_{XX}(w) \geq 0$. (207)

2. $G_{XX}(-w) = G_{XX}(w)$ for real $X(t)$. (208)

3. $G_{XX}(w)$ is real. (209)

4. $P_{XX} = \frac{1}{2\pi} \int_{-\infty}^{\infty} G_{XX}(w)\, dw$. (210)

5. $G_{XX}(w) = \int_{-\infty}^{\infty} R_{XX}(\tau)\, e^{-jw\tau}\, d\tau$. (211)

6. $R_{XX}(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} G_{XX}(w)\, e^{jw\tau}\, dw$. (212)

The power spectral density $G_{XX}(w)$ of a stationary random process $X(t)$ is the Fourier transform of the auto-correlation function of $X(t)$.

Passage of a Random Process through a Linear Time-Invariant Filter

Figure 6

Suppose that an ergodic random process $X(t)$ with a power spectral density $G_{XX}(f)$ is fed through a linear time-invariant filter whose transfer function is $H(f)$ and whose impulse-response is $h(t)$. Let the output signal from the filter be $Y(t)$ with a power spectral density $G_{YY}(f)$. The auto-correlation function is

$$R_{YY}(\tau) = E[Y(t)Y(t+\tau)] \tag{213}$$

and $Y(t)$ is the convolutional of $X(t)$ and $h(t)$

$$Y(t) = \int_{-\infty}^{\infty} X(t-\varepsilon_1)h(\varepsilon_1)d\varepsilon_1 \tag{214}$$

Also,

$$Y(t+\tau) = \int_{-\infty}^{\infty} X(t+\tau-\varepsilon_2)h(\varepsilon_2)d\varepsilon_2 \tag{215}$$

Substitute equations (214) and (215) into eqn. (213), we get

$$R_{YY}(\tau) = E[\int_{-\infty}^{\infty} X(t-\varepsilon_1)h(\varepsilon_1)d\varepsilon_1 \int_{-\infty}^{\infty} X(t+\tau-\varepsilon_2)h(\varepsilon_2)d\varepsilon_2] \tag{216}$$

Since the averaging is linear, we can interchange the averaging and integration operations. Therefore,

$$R_{YY}(\tau) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} E[X(t-\varepsilon_1)X(t-\varepsilon_2)]h(\varepsilon_1)h(\varepsilon_2)\, d\varepsilon_1 d\varepsilon_2 \tag{217}$$

Furthermore,

$$R_{XX}(\tau) = E[X(t)X(t+\tau)]$$
$$R_{XX}(\tau) = E[X(t-\varepsilon_1)X(t+\tau-\varepsilon_1)]$$
$$R_{XX}(\varepsilon_1-\varepsilon_2) = E[X(t-\varepsilon_1)X(t-\varepsilon_1)]$$
$$R_{XX}[\varepsilon_1-(\varepsilon_2-\tau)] = E\{X(t-\varepsilon_1)X[t-(\varepsilon_2-\tau)]\}$$
$$R_{XX}[\varepsilon_1-(\varepsilon_2-\tau)] = E[X(t-\varepsilon_1)X(t+\tau-\varepsilon_2)] \tag{218}$$

Substitute eqn. (218) into eqn. (217),

$$R_{YY}(\tau) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_{XX}[\varepsilon_1-(\varepsilon_2-\tau)]h(\varepsilon_1)h(\varepsilon_2)\, d\varepsilon_1 d\varepsilon_2$$

$$R_{YY}(\tau) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_{XX}(\tau+\varepsilon_1-\varepsilon_2)h(\varepsilon_1)h(\varepsilon_2) d\varepsilon_1 d\varepsilon_2$$

$$G_{YY}(f) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_{XX}(\tau+\varepsilon_1-\varepsilon_2)h(\varepsilon_1)h(\varepsilon_2)\, e^{-j2\pi f\tau} d\varepsilon_1 d\varepsilon_2 d\tau \tag{219}$$

Let $\varepsilon = \tau + \varepsilon_1 - \varepsilon_2$, we have $d\varepsilon = d\tau$

$$G_{YY}(f) =$$

$$\int_{-\infty}^{\infty} h(\varepsilon_1)e^{j2\pi f\varepsilon_1} d\varepsilon_1 \int_{-\infty}^{\infty} h(\varepsilon_2)e^{-j2\pi f\varepsilon_2} d\varepsilon_2 \int_{-\infty}^{\infty} R_{XX}(\varepsilon)e^{-j2\pi f\varepsilon} d\varepsilon$$

$$G_{YY}(f) = H^*(f)\, H(f)\, G_{XX}(f) \tag{220}$$

$$G_{YY}(f) = |H(f)|^2\, G_{XX}(f) \tag{221}$$

If $X(t)$ is a zero mean Gaussian random process, it can be shown that $Y(t)$ is also a zero mean Gaussian random process, regardless of the transfer function of the filter. If $X(t)$ is a white (mean = 0) Gaussian noise, it will have a two-sided noise power spectral density of $G_{NN}(f) = N_0/2$.

White Gaussian noise is a zero-mean Gaussian random process with a uniform power density spectrum over all frequencies. For coloured Gaussian noise, the power spectral density is not flat over the entired bandwidth. Suppose the input to the filter is coloured Gaussian noise, such that its two-sided power spectral density is $G_{CC}(f)$, this coloured noise can be first be filtered through a noise-whitening filter whose transfer function is $\sqrt{(N_0/2)(G_{CC}(f)}$ and the overall response of the model can be analysed under white noise disturbance. Whitening filter is used because we are often encountered practical systems with additive white Gaussian noise (AWGN).

Further Analysis of White Noise

White Gaussian noise has a uniform power spectral density distributed over a very wide range of frequencies up to about 10^{13} Hz. The two-sided power spectral density $G_{XX}(f)$ is equal to $N_0/2$. Hence, we obtain for the auto-correlation function from $G_{XX}(f)$, where

$$R_{XX}(\tau) = \int_{-\infty}^{\infty} G_{XX}(f)\, e^{j2\pi f\tau}\, df \tag{222}$$

$$R_{XX}(\tau) = N_0/2 \int_{-\infty}^{\infty} e^{j2\pi f\tau}\, df$$

$$R_{XX}(\tau) = (N_0/2)\, \delta(\tau).$$

Since $R_{XX}(\tau)$ has a value at $\tau = 0$ only, there is no correlation between any two samples of white noise separated by an interval $\tau > 0$. For uncorrelated Gaussian noise samples, they are therefore statistically independent.

Figure 7(a)

Figure 7(b)

The average power which is given by

$$R_{XX}(\tau = 0) = \int_{-\infty}^{\infty} G_{XX}(f)\, df$$

$$R_{XX}(\tau = 0) = \infty$$

Coloured Noise

Most communication systems are band-limited, a bandlimited white Gaussian noise is called a coloured noise.

Consider the case where a white noise is passed through an ideal low-pass filter with a bandwidth +/- B Hz and a transfer function $|H(f)| = 1$. The filter output is

$$G_{YY}(f) = |H(f)|^2\, G_{XX}(f). \tag{223}$$

The auto-correlation function $R_{YY}(\tau)$ of the filtered white noise is

$$R_{YY}(\tau) = \int_{-\infty}^{\infty} G_{YY}(f)\, e^{j2\pi f\tau}\, df$$

$$R_{YY}(\tau) = \int_{-B}^{B} G_{YY}(f)\, e^{j2\pi f\tau}\, df$$

$$R_{YY}(\tau) = N_0/2 \int_{-B}^{B} e^{j2\pi f\tau}\, df$$

$$R_{YY}(\tau) = N_0 B \, \frac{\sin(2\pi B\tau)}{2\pi B\tau} \tag{224}$$

Figure 8(a)

Figure 8(b)

The average power of the filtered noise is

$$R_{YY}(\tau = 0) = \int_{-B}^{B} G_{YY}(f)\, df$$

$$R_{YY}(\tau = 0) = N_0 B \quad \text{watts.}$$

Since the shape of $R_{YY}(\tau)$ is a sinc function, a filtered white noise produces correlated, with periodicity, bandlimited white noise.

References

[1] Schanmugan, K. S., 'Digital and Analog Communication Systems', John Wiley & Sons, 1979.

[2] Peebles, P. Z., 'Probability, Random Variables and Random Signal Principles', 2/e, McGraw-Hill, 1987.

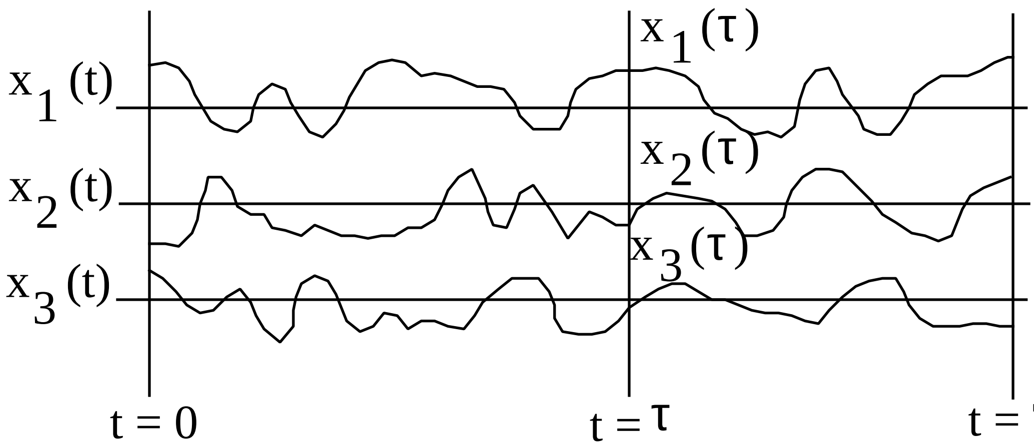

Figure 5

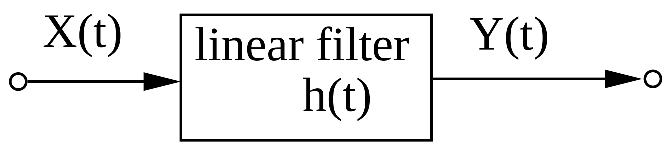

Figure 6

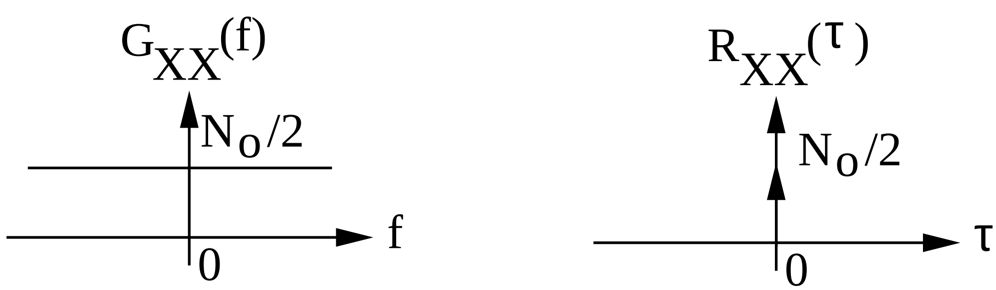

Figure 7(a) **Figure 7(b)**

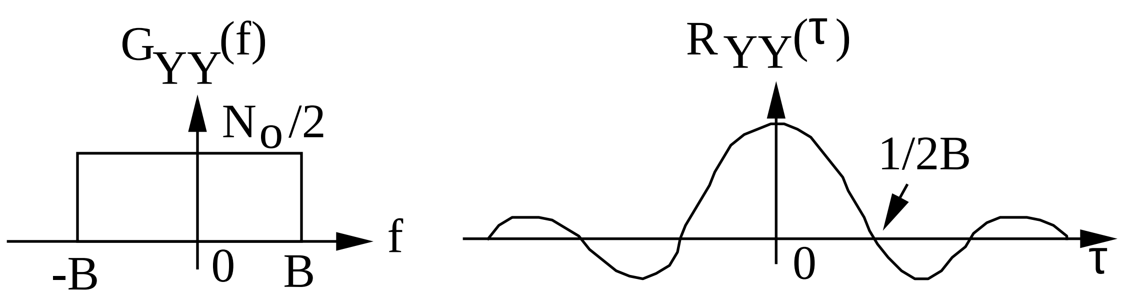

Figure 8(a) **Figure 8(b)**