37. Syndrome Decoding and Performance of Block Codes

Syndrome Decoding

Suppose that a code vector $\mathbf{V} = [v_0\ v_1\ ...\ v_{n-1}]$ is transmitted. In the presence of noise, the received vector $\mathbf{R} = [r_0\ r_1\ ...\ r_{n-1}]$ may not be the same as the transmitted vector. The decoder computes the $(n\text{-}k)$-tuple :

$$\mathbf{S} = \mathbf{R}\ \mathbf{H}^T \tag{37.1}$$

where $\mathbf{H}^T$ is the transpose of the parity-check matrix of the (n, k) linear code and $\mathbf{S}$ is the syndrome of $\mathbf{R}$. In the presence of errors,

$$\mathbf{R} = \mathbf{V} + \mathbf{E} \tag{37.2}$$

where $\mathbf{R} = [r_0\ r_1\ ...\ r_{n-1}]$ is the received word, $\mathbf{V} = [v_0\ v_1\ ...\ v_{n-1}]$ is the transmitted codeword, and $\mathbf{E} = [e_0\ e_1\ ...\ e_{n-1}]$ is the error pattern.

For (n, k) systematic linear code with an information vector $\mathbf{U} = [u_0\ u_1\ ...\ u_{k-1}]$, the transmitted codeword $\mathbf{V}$ becomes

$$\mathbf{V} = [u_0\ u_1\ ...\ u_{k-1}\ v_k\ \ v_{k+1}\ ...\ v_{n-1}\ ] \tag{37.3}$$

and equation (37.1) becomes

$$\mathbf{S} = \mathbf{R}\ \mathbf{H}_{\mathrm{SEF}}{}^T, \tag{37.4}$$

where

$$\mathbf{S} = [s_0\ s_1\ ...\ s_{n-k-1}]. \tag{37.5}$$

and

$$\mathbf{H}_{SEF} = \begin{bmatrix} -p_{0,0} & -p_{1,0} & \cdots & -p_{k-1,0} & 1 & 0 & \cdots & 0 \\ -p_{0,1} & -p_{1,1} & \cdots & -p_{k-1,1} & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ -p_{0,n-k-1} & -p_{1,n-k-1} & \cdots & -p_{k-1,n-k-1} & 0 & 0 & \cdots & 1 \end{bmatrix} \qquad (37.6)$$

Based on equations (37.5) and (37.6), equation (37.4) can be written as,

$$\begin{aligned} s_0 &= r_k + (-r_0 p_{0,0} - r_1 p_{1,0} - \dots - r_{k-1} p_{k-1,0}) \\ s_1 &= r_{k+1} + (-r_0 p_{0,1} - r_1 p_{1,1} - \dots - r_{k-1} p_{k-1,1}) \\ &: \qquad\qquad\qquad : \\ s_{n-k-1} &= r_{n-1} + (-r_0 p_{0,n-k-1} - r_1 p_{1,n-k-1} \\ &\quad - \dots - r_{k-1} p_{k-1,n-k-1}) \end{aligned} \qquad (37.7)$$

It can be seen that the first term corresponds to the received parity-check digit and the remaining terms correspond to the re-calculated parity-check digits.

The syndrome is the sum of the received parity-check digits and the recomputed parity-check digits based on the received digits $r_0, r_1, \dots, r_{k-1}$. If the syndrome vector $\mathbf{S} = \mathbf{0}$, then it is assumed that no error has occurred. If the syndrome is non-zero, the presence of errors has been detected. Depending on the transmitted format of the codeword, the error locations can be identified accordingly.

Since the syndrome vector $\mathbf{S} = \mathbf{R}\,\mathbf{H}^T$, $\mathbf{R} = \mathbf{V} + \mathbf{E}$ and $\mathbf{V}\,\mathbf{H}^T = \mathbf{0}$, it follows that

$$\mathbf{S} = \mathbf{E}\,\mathbf{H}^T \qquad (37.8)$$

Decoding can be accomplished by a table look-up in the following manner :

1. Compute the syndrome $\mathbf{S} = \mathbf{R}\,\mathbf{H}^T$.

2. If the syndrome $\mathbf{S}$ is zero, we assume that $\mathbf{R}$ is error free.

3. If $\mathbf{S}$ is non-zero, we assume that $\mathbf{R}$ contains errors. The error pattern that corresponds to $\mathbf{S}$ is subtracted from the received vector for error correction.

Consider a (5, 2) single-error-correcting binary linear block code. The generator matrix, as given, is $\mathbf{G} = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \end{bmatrix}$ and the parity-check matrix of the code is $\mathbf{H} = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \end{bmatrix}$. If we choose the same set of error patterns to compute the syndromes for decoding, the syndrome decoding table is given in Table 37.1.

Table 37.1
Syndrome Decoding Table for the (5, 2) Binary Linear Block Code

S (*Binary-Coded-Decimal*)	**E** (*Binary-Coded-Decimal*)
0 0 1 (1)	0 0 0 0 1 (1)
0 1 0 (2)	0 0 0 1 0 (2)
1 0 0 (4)	0 0 1 0 0 (4)
1 1 1 (7)	0 1 0 0 0 (8)
0 1 1 (3)	1 0 0 0 0 (16)
1 0 1 (5)	0 0 1 0 1 (5)
1 1 0 (6)	0 0 1 1 0 (6)

Suppose the transmitted code vector is **V** = [0 1 1 1 1] and the error vector is **E** = [0 0 0 0 1]. The received vector is **R** = **V** + **E** = [0 1 1 1 0]. The syndrome vector is **S** = **R** $\mathbf{H}^T$ = [0 0 1] and **E** = [0 0 0 0 1] is taken as the estimated error vector. The decoded vector is $\hat{\mathbf{V}}$ = **R** + **E** = [0 1 1 1 1]. Single error is corrected. Next, consider the same transmitted code vector **V**. Suppose the error vector is **E** = [0 0 0 1 1]. The received vector is **R** = [0 1 1 0 0]. The syndrome vector is **S** = **R** $\mathbf{H}^T$ = [0 1 1] and **E** = [1 0 0 0 0] is taken as the estimated error vector. The decoded vector is $\hat{\mathbf{V}}$ = **R** + **E** = [1 1 1 0 0]. Double errors cannot be corrected because the code is a single-error-correcting code.

The general form of encoder and decoder for an (n, k) systematic binary linear block code is shown in Figure 37.1.

Figure 37.1 (a) (n, k) systematic binary linear block encoder, and (b) syndrome decoder.

Performance of Binary Block Codes

If the minimum Hamming distance of a block code is d_{min}, any two distinct codewords will differ in at least d_{min} places. No error pattern of d_{min} - 1 or less can change one codeword into another codeword after transmission. As a result,

1. The code can detect all error patterns of $d_{min} - 1$ or less.

2. An (n, k) binary linear code can detect $2^n - 2^k$ error patterns of length n.

3. There are $2^k - 1$ undetectable error patterns. This is due to the fact that there are exactly 2^k-1 non-zero error patterns that are identical to the 2^k-1 non-zero codewords. They can change a transmitted codeword to another codeword.

4. If the code is used for error detection only on a binary symmetric channel with minimum hard-decision distance decoding. The decoder fails to detect the presence of errors whenever an error pattern changes the transmitted codeword to another codeword. The probability of undetected word error is

$$P_u = \sum_{i=1}^{n} A_i\, p'^i\, (1-p')^{n-i} \qquad (37.9)$$

where p' is the channel transition probability, A_i is the number of codewords of weight i in the code, and $A_1 = A_2 = \ldots = A_{d_{min}-1} = 0$. Equation (37.9) can be written as

$$P_u = \sum_{i=d_{min}}^{n} A_i\, p'^i\, (1-p')^{n-i} \qquad (37.10)$$

5. If the code is used for error correction only on a binary symmetric channel with minimum hard-decision distance decoding, the probability of decoding word error is upper bounded by

$$P_e \leq \sum_{i=t'+1}^{n} \binom{n}{i} p'^i\, (1-p')^{n-i}. \qquad (37.11)$$

where $\binom{n}{i}$ gives the number of error patterns of weight i and $p'^{i}(1-p')^{n-i}$ is the probability of occurrence of a particular weight i error pattern.

Example 37.1

The codewords of an (4, 1) binary repetition code are 0 0 0 0 and 1 1 1 1. The total number of undetectable error patterns is $2^k-1 = 1$ and the total number of detectable error patterns $= 2^n-2^k = 14$. The probability of undetected word error is

$$P_u = \sum_{i=1}^{n=4} A_i\, p'^{i}(1-p')^{n-i}$$

$$= A_1 p'^1(1-p')^3 + A_2 p'^2(1-p')^2 + A_3 p'^3(1-p')^1 + A_4 p'^4(1-p')^0$$

$$= 0 + 0 + 0 + p'^4(1-p')^0.$$

The probability of decoding word error is

$$P_e \leq \sum_{i=t'+1}^{n=4} \binom{n}{i} p'^{i}(1-p')^{n-i}$$

$$\leq \binom{4}{2} p'^2(1-p')^2 + \binom{4}{3} p'^3(1-p')^1 + \binom{4}{4} p'^4(1-p')^0$$

$$\leq 6p'^2(1-p')^2 + 4p'^3(1-p')^1 + 1p'^4(1-p')^0.$$

For $p' = 0.01$, the decoding word error probability on a binary symmetric channel is

$$P_e \leq 0.0011800899.$$

Let the probability of selecting an incorrect codeword be P_d. For coherent BPSK signals with AWGN channels and unquantised soft-decision decoding, it can be shown that

$$P_d = Q\left(\sqrt{2d_{\min}(k/n)E_b/N_o}\right) \qquad (37.12)$$

where

$$Q(\alpha'') = \frac{1}{\sqrt{2\pi}} \int_{\alpha''}^{\infty} e^{-\beta''^2/2} d\beta'' \quad (37.13)$$

and E_b/N_0 is the ratio of average bit energy to noise power-spectral-density. For all incorrectly selected codewords, the decoding word error probability is upper bounded by

$$P_e \leq \sum_{i=1}^{n} A_i P_d \quad (37.14)$$

and

$$P_e \leq \sum_{i=1}^{n} A_i Q\left(\sqrt{2i(k/n)E_b/N_o}\right) \quad (37.15)$$

Reference

Lee, L. H. C., Error-Control Block Codes for Communications Engineers, Artech House, 2000.

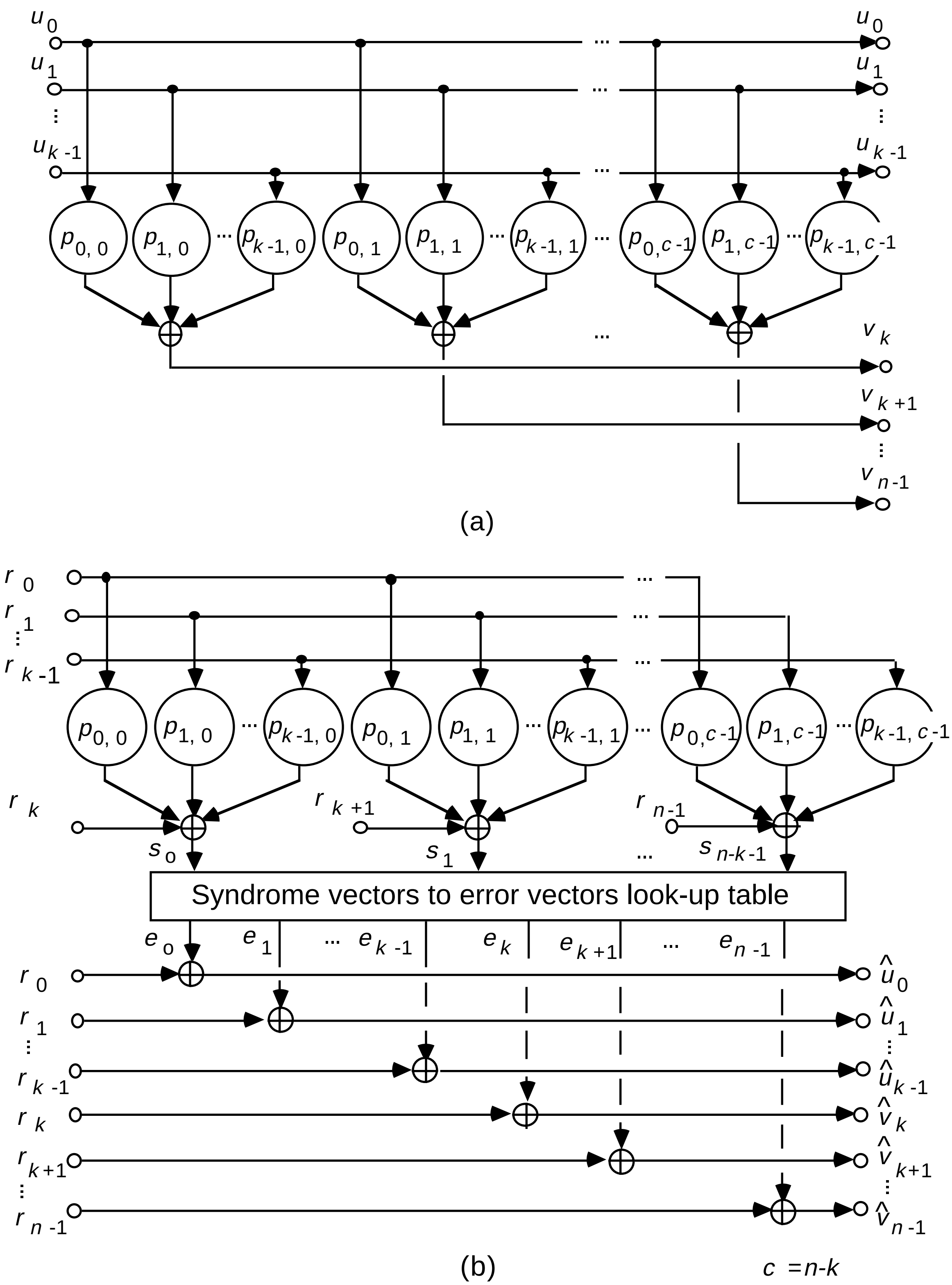

Figure 37.1 (a) (n, k) systematic binary linear block encoder, and (b) syndrome decoder.