

## 26. Performance of Binary Baseband Signals in the Presence of Noise

### Error Performance of Unipolar Signal (NRZ)

We are here concerned with the detection of some given parameters of a sampled baseband binary signal in the presence of noise. The received signal is assumed to be the continuous baseband waveform

$$r(t) = s(t) + n(t) \quad (26.1)$$

where  $s(t)$  is the waveform of the binary signal and  $n(t)$  is the noise waveform. Assume that the noise is a **white Gaussian noise (WGN)** and that the **probability density function (pdf)** of the noise sample  $n$  is

$$f(n) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-n^2/2\sigma^2} \quad (26.2)$$

where  $\sigma^2$  is the variance of the noise. This is shown in Figure 26.1.

**Figure 26.1** Gaussian probability density function.

The driving force behind this Gaussian model for the noise is the **Central Limit Theorem** which states that the sum of a number of random variables has a Gaussian probability density function.

### **Figure 26.2** Effect of noise in the transmission of unipolar NRZ signals

Consider the effect of noise in the transmission of a unipolar NRZ signal as shown in Figure 26.2. If we sample in the middle of each signal pulse of duration  $T$  seconds, the sampled output consists of a signal  $A$  plus noise or a noise alone. These samples are random variables and have the same statistics as the noise samples. Figure 26.3 shows the conditional probability densities in the transmission of unipolar NRZ signals.

**Figure 26.3** Conditional probability densities in the transmission of unipolar NRZ signals.

The conditional probability density function of  $u$  assuming a 0 is sent is

$$f(u/0) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{u^2}{2\sigma^2}} \quad (26.3)$$

and the probability of error given a 0 is sent is

$$P_{e0} = P(u > A/2) = \int_{A/2}^{\infty} f(u/0) du = \int_{A/2}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{u^2}{2\sigma^2}} du \quad (26.4)$$

Similarly, the probability of error given a 1 is sent is

$$P_{e1} = P(u < A/2) = \int_{-\infty}^{A/2} f(u/1) du = P_{e0} \quad (26.5)$$

where

$$f(u/1) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(u-A)^2}{2\sigma^2}} \quad (26.6)$$

Let  $p_0$  be the probability of sending a 0 and  $p_1$  be the probability of sending a 1. For equally likely transmission of binary signals, we have  $p_0 = p_1 = 0.5$ . The average probability of error is

$$\begin{aligned} P_e &= p_0 P_{e0} + p_1 P_{e1} \\ &= P_{e0} \end{aligned} \quad (26.7)$$

Let  $y = \frac{u}{\sqrt{2}\sigma}$ . Then  $y^2 = \frac{u^2}{2\sigma^2}$  and  $dy = \frac{du}{\sqrt{2}\sigma}$ . Substituting  $y$  and  $dy$  into equation (26.4), we get

$$\begin{aligned} P_e &= \int_{y=u/\sqrt{2}\sigma}^{\infty} \frac{1}{\sqrt{\pi}} e^{-y^2} dy \\ &= \frac{1}{\sqrt{\pi}} \int_{y=u/\sqrt{2}\sigma}^{\infty} e^{-y^2} dy \\ &= \frac{1}{2} \left[ \frac{2}{\sqrt{\pi}} \int_{u/\sqrt{2}\sigma}^{\infty} e^{-y^2} dy \right] \end{aligned} \quad (26.8)$$

This integral expression inside the square brackets can only be evaluated by numerical integration; tables of the Gaussian integral exist. It is called the *error function*  $\text{erf}(x)$  and is defined as

$$\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-\alpha^2} d\alpha \quad (26.9)$$

If the probability of error is small, it is more convenient to compute the *complementary error function*

$$\begin{aligned} \text{erfc}(x) &= 1 - \text{erf}(x) \\ &= \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-\alpha^2} d\alpha \end{aligned} \quad (26.10)$$

Figure 26.4 shows a plot of the complementary function.

**Figure 26.4** Complementary error function.

Substituting equation (26.10) into equation (26.8), we have

$$\begin{aligned} P_e &= \frac{1}{2} \text{erfc} \left( \frac{u}{\sqrt{2}\sigma} \right) \\ &= \frac{1}{2} \text{erfc} \left( \frac{A}{2\sqrt{2}\sigma} \right) \end{aligned} \quad (26.11)$$

where  $u = A/2$  for threshold. The average probability of error depends only on the *signal amplitude*  $A$  and  $\sigma$ , the *rms value of the noise*.

### Error Performance of Polar Signal (NRZ)

In the polar NRZ signal case, the sampled output consists of a polar signal  $\pm A$  plus noise. Figure 26.5 shows the conditional probability densities in the transmission of polar NRZ signals.

**Figure 26.5** Conditional probability densities in the transmission of polar NRZ signals.

The conditional probability density function of  $u$  assuming a 0 is sent is

$$f(u/0) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(u+A)^2}{2\sigma^2}} \quad (26.12)$$

and the probability of error given a 0 is sent is

$$P_{e0} = P(u > 0) = \int_0^{\infty} f(u/0) du = \int_0^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(u+A)^2}{2\sigma^2}} du \quad (26.13)$$

Similarly, the probability of error given a 1 is sent is

$$P_{e1} = P(u < 0) = \int_{-\infty}^0 f(u/1) du = P_{e0} \quad (26.14)$$

where

$$f(u/1) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(u-A)^2}{2\sigma^2}} \quad (26.15)$$

Let  $p_0$  be the probability of sending a 0 and  $p_1$  be the probability of sending a 1. For equally likely transmission of binary signals, we have  $p_0 = p_1 = 0.5$ . The average probability of error is

$$\begin{aligned} P_e &= p_0 P_{e0} + p_1 P_{e1} \\ &= P_{e0} \end{aligned} \quad (26.16)$$

Let  $y = \frac{u+A}{\sqrt{2\sigma}}$ . Then  $y^2 = \frac{(u+A)^2}{2\sigma^2}$  and  $dy = \frac{du}{\sqrt{2\sigma}}$ . Substituting  $y$  and  $dy$  into equation (26.13), we get

$$\begin{aligned} P_e &= \int_{y=\frac{u+A}{\sqrt{2\sigma}}}^{\infty} \frac{1}{\sqrt{\pi}} e^{-y^2} dy \\ &= \frac{1}{2} \frac{2}{\sqrt{\pi}} \int_{y=\frac{u+A}{\sqrt{2\sigma}}}^{\infty} e^{-y^2} dy \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2} \operatorname{erfc} \left( \frac{u+A}{\sqrt{2}\sigma} \right) \\
&= \frac{1}{2} \operatorname{erfc} \left( \frac{A}{\sqrt{2}\sigma} \right)
\end{aligned} \tag{26.17}$$

where  $u = 0$  for threshold. Again, the average probability of error depends only on the signal amplitude  $A$  and  $\sigma$ .

Table 26.1 summarises the performance of polar NRZ and unipolar NRZ signals due to AWGN.

Baseband signal	$P_e$	Relative dB
Polar NRZ	$\frac{1}{2} \operatorname{erfc} \left( \frac{A}{\sqrt{2}\sigma} \right)$	0
Unipolar NRZ	$\frac{1}{2} \operatorname{erfc} \left( \frac{A}{2\sqrt{2}\sigma} \right)$	-3

**Table 26.1** Performance of polar NRZ and unipolar NRZ signals in the presence of AWGN.

### Choice of Decision Threshold Levels

So far, we have arbitrarily chosen the decision level as  $A/2$  for unipolar NRZ signals and 0 for polar NRZ signals. One may ask the following question. Is it possible to further reduce the error probability by choosing some other decision level? Consider the conditional probability density functions in the transmission of polar NRZ signals as shown in Figure 26.6.

**Figure 26.6** Conditional probability densities in the transmission of polar NRZ signals with choice of decision level.

From our previous work, we know that the average probability of error is given by

$$\begin{aligned}
P_e &= p_0 P_{e0} + p_1 P_{e1} \\
&= p_0 \int_d^{\infty} f(u/0) du + p_1 \int_{-\infty}^d f(u/1) du
\end{aligned} \tag{26.18}$$

To find the optimum decision level, we simply differentiate  $P_e$  with respect to  $u$  and set it equal to zero. For  $u = d$ , we have

$$\left. \frac{\partial P_e}{\partial u} \right|_{u=d} = -p_0 f(d/0) + p_1 f(d/1) = 0$$

and

$$\frac{p_0}{p_1} = \frac{f(d/0)}{f(d/1)} \quad (26.19)$$

It can be seen that the optimum value of  $d$  depends on  $f(d/0)$ ,  $f(d/1)$ ,  $p_0$  and  $p_1$ . For the transmission of polar NRZ signals in the presence of AWGN, the conditional probability density functions are given by

$$f(d/0) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(d+A)^2}{2\sigma^2}}$$

and

$$f(d/1) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(d-A)^2}{2\sigma^2}}$$

Therefore

$$\begin{aligned} \frac{p_0}{p_1} &= \frac{e^{-\frac{(d-A)^2}{2\sigma^2}}}{e^{-\frac{(d+A)^2}{2\sigma^2}}} \\ &= e^{\frac{(d+A)^2 - (d-A)^2}{2\sigma^2}} \\ &= e^{2dA/\sigma^2} \end{aligned}$$

and

$$d = \frac{\sigma^2}{2A} \ln \frac{p_0}{p_1} \quad (26.20)$$

In practice, the choice of optimum  $d$  is not particularly critical. A shift away from the usual half-way threshold line is only a fraction of the rms value of the noise, an insignificant change. Why discuss it? There are several reasons behind this issue.

1. At least, we know that the half-way threshold line is close enough to give near optimum system performance.
2. If the noise is not Gaussian, we need to determine the optimum value of  $d$ .

### **Reference**

- [1] M. Schwartz, Information Transmission, Modulation, and Noise, 4/e, McGraw-Hill, 1990.

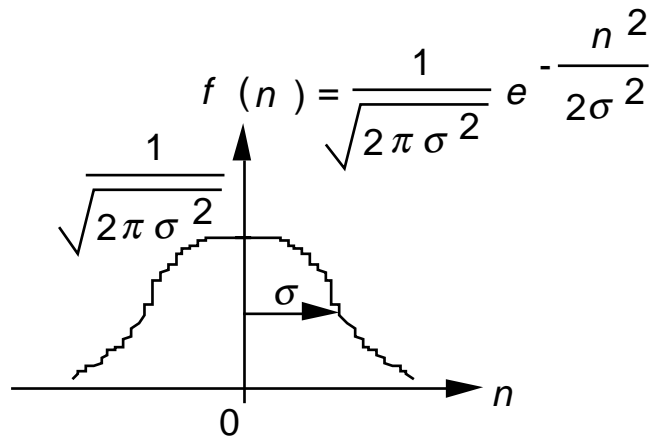


Figure 26.1 Gaussian probability density function.

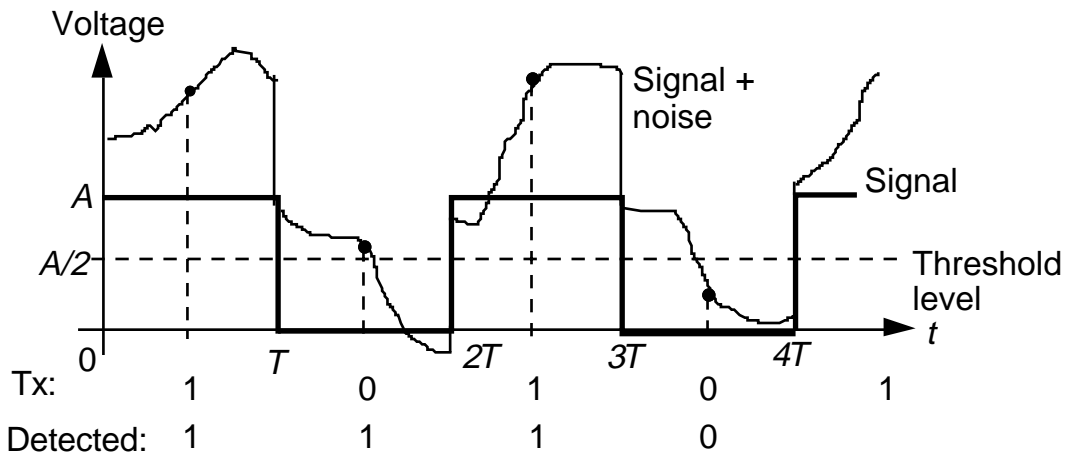


Figure 26.2 Effect of noise in the transmission of unipolar NRZ signals.

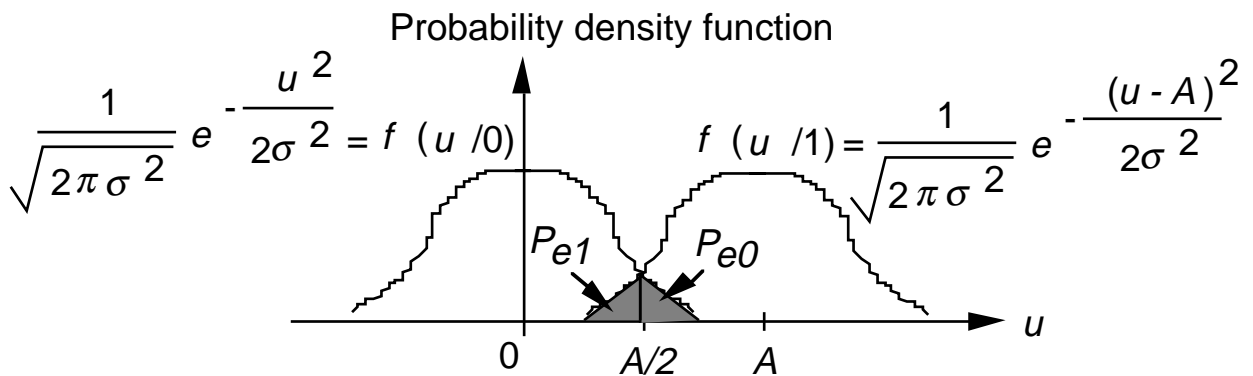


Figure 26.3 Conditional probability densities in the transmission of unipolar NRZ signals.

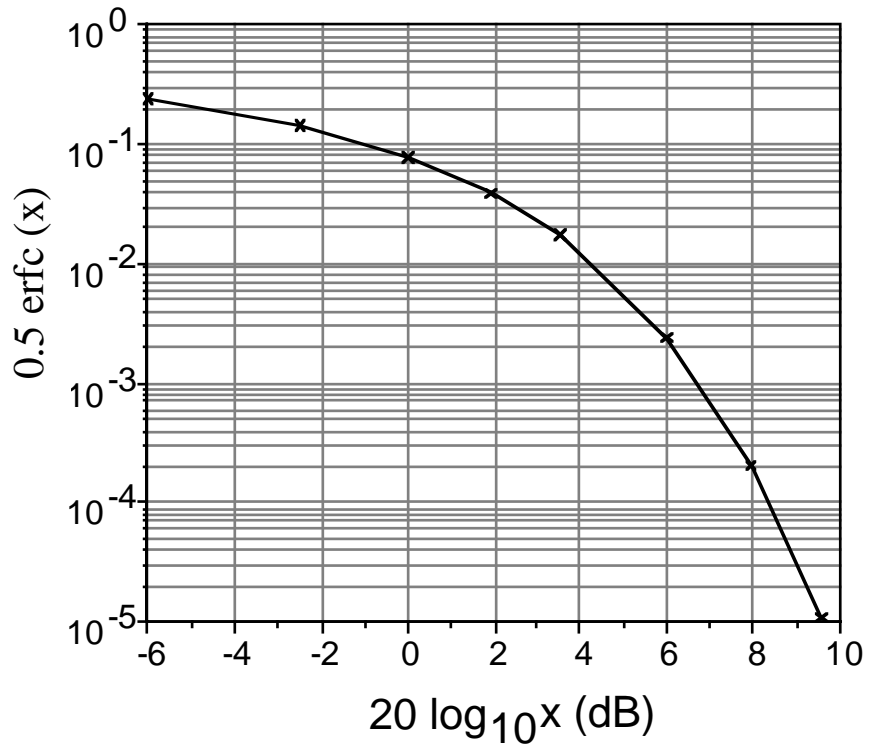


Figure 26.4 Complementary error function.

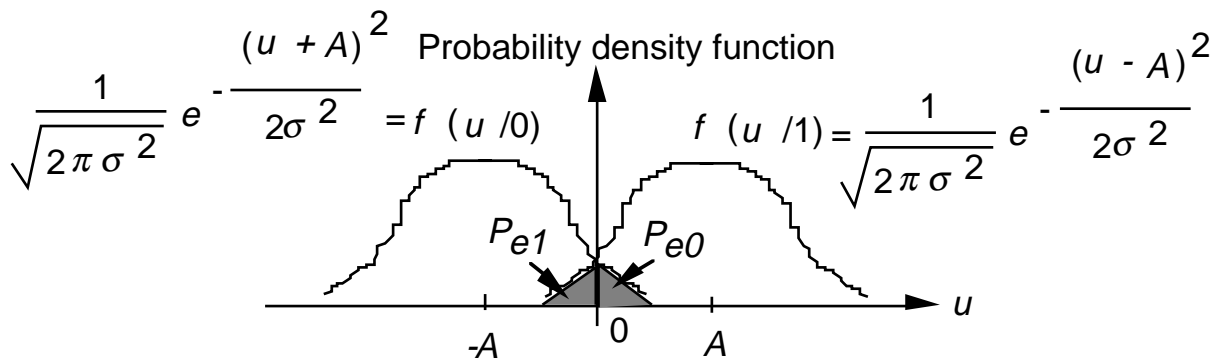


Figure 26.5 Conditional probability densities in the transmission of polar NRZ signals.

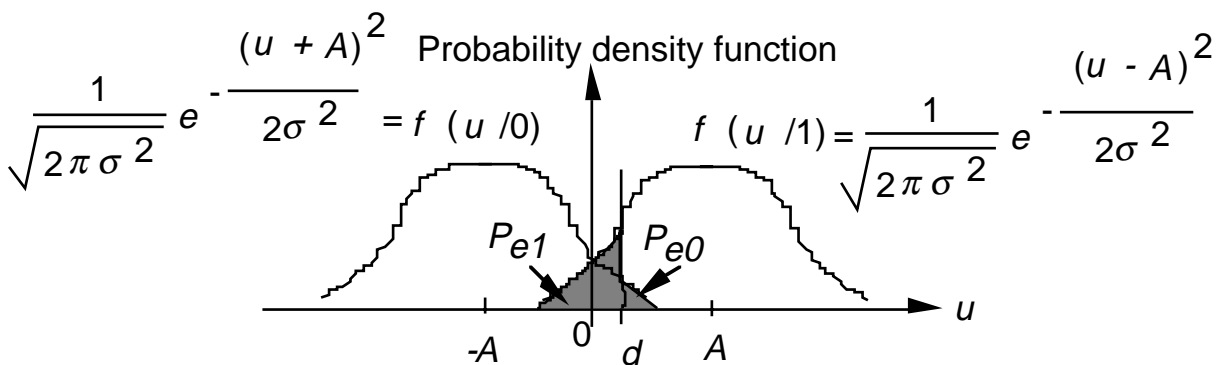


Figure 26.6 Conditional probability densities in the transmission of polar NRZ signals with choice of decision level.