2. Review of Fourier Series

The Time-Frequency Concept [1]

Consider the following set of time functions $s(t) = \{3A \sin \omega_0 t, A \sin 2\omega_0 t\}$. We can represent these functions in different ways by plotting the amplitude versus time t, amplitude versus angular frequency ω, or amplitude versus frequency f.

Figure 2.1 (a) Amplitude-time plot, and (b) amplitude-angular frequency plot.

$\omega_0 = 2\pi/T_0$ is called the *fundamental angular frequency* and $\omega_2 = 2\omega_0$ is called the second *harmonic* of the fundamental. In general, $\omega_n = n\omega_0$ is said to be the nth harmonic of the fundamental, where $n > 1$.

In communication engineering we are interested in *steady-state* analysis much of the time.

The Fourier series provides a useful model for analysing the frequency content and the steady-state network response for periodic input signals.

Trigonometric (Quadrature) Fourier Series [1]

A *periodic* time function $s(t)$ over the interval $a - \frac{T_0}{2} < t < a + \frac{T_0}{2}$ may be represented by an infinite sum of sinusoidal waveforms

$$s(t) = \frac{a_0}{T_0} + \frac{2}{T_0} \sum_{n=1}^{\infty} (a_n \cos n\omega_0 t + b_n \sin n\omega_0 t) \qquad (2.1)$$

where T_0 is the *period* of the *fundamental frequency* f_0 and $f_0 = 1/T_0$. This is called the *trigonometric (quadrature) Fourier series* representation of the time function $s(t)$. The coefficients a_n and b_n are given by

$$a_n = \int_{a-T_0/2}^{a+T_0/2} s(t) \cos n\omega_0 t \, dt, \; n \geq 0 \qquad (2.2)$$

and

$$b_n = \int_{a - T_0/2}^{a + T_0/2} s(t) \sin n\omega_0 t \, dt, \; n > 0 \qquad (2.3)$$

The choice of a is arbitrary, and it is usually set to 0.

Many forms of the trigonometric Fourier series may be written. For example,

$$s(t) = a'_0 + \sum_{n=1}^{\infty} (a'_n \cos n\omega_0 t + b'_n \sin n\omega_0 t) \qquad (2.4)$$

is commonly used. The coefficients a'_n and b'_n are given by

$$a'_n = \frac{2}{T_0} \int_{a - T_0/2}^{a + T_0/2} s(t) \cos n\omega_0 t \, dt, \; n \geq 0 \qquad (2.5)$$

and

$$b'_n = \frac{2}{T_0} \int_{a - T_0/2}^{a + T_0/2} s(t) \sin n\omega_0 t \, dt, \; n > 0 \qquad (2.6)$$

Example 2.1 Find the trigonometric Fourier series for the periodic time function $s(t)$ shown in Figure 2.2.

Figure 2.2 A periodic rectangular waveform.

$$a_n = \int_{-T_0/2}^{T_0/2} s(t) \cos n\omega_0 t \, dt$$

$$a_n = \int_{-\tau/2}^{\tau/2} A_m \cos n\omega_0 t \, dt$$

$$a_n = 2A_m \frac{\sin n\omega_0 \tau/2}{n\omega_0} = A_m \tau \frac{\sin n\omega_0 \tau/2}{n\omega_0 \tau/2}$$

$$b_n = \int_{-T_0/2}^{T_0/2} s(t) \sin n\omega_0 t \, dt = 0$$

Therefore, $s(t) = \frac{A_m \tau}{T_0} + \frac{2}{T_0} \sum_{n=1}^{\infty} \left(A_m \tau \frac{\sin n\omega_0 \tau/2}{n\omega_0 \tau/2}\right) \cos n\omega_0 t$.

Exponential (Complex or Phasor) Fourier Series [1]

The time function $s(t)$ may be represented over the interval $a - \frac{T_0}{2} < t < a + \frac{T_0}{2}$ by the equivalent *exponential (complex or phasor) Fourier series*

$$s(t) = \frac{1}{T_0} \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t} \tag{2.7}$$

where the coefficients c_n are given by

$$c_n = \int_{a-T_0/2}^{a+T_0/2} s(t) e^{-jn\omega_0 t} dt \tag{2.8}$$

c_0 is equivalent to the dc value of the waveform $s(t)$.
c_n is, in general, a complex number. Furthermore, it is a phasor since it is the coefficient of $e^{jn\omega_0 t}$.

The complex Fourier series is easier to use for analytical problems.

Many forms of the complex Fourier series may be written. For example,

$$s(t) = \sum_{n=-\infty}^{\infty} c'_n e^{jn\omega_0 t} \tag{2.9}$$

is commonly used. The coefficients c'_n are given by

$$c'_n = \frac{1}{T_0} \int_{a-T_0/2}^{a+T_0/2} s(t) e^{-jn\omega_0 t} dt \tag{2.10}$$

Example 2.2 Find the complex Fourier series for the periodic time function $s(t)$ shown in Figure 2.2.

$$c_n = \int_{-T_0/2}^{T_0/2} s(t)\, e^{-jn\omega_0 t}\, dt$$

$$c_n = \int_{-\tau/2}^{\tau/2} A_m\, e^{-jn\omega_0 t}\, dt$$

$$c_n = A_m \frac{e^{jn\omega_0\tau/2} - e^{-jn\omega_0\tau/2}}{jn\omega_0}$$

$$c_n = 2A_m \frac{\sin n\omega_0\tau/2}{n\omega_0} = A_m\tau \frac{\sin n\omega_0\tau/2}{n\omega_0\tau/2}$$

Therefore, $s(t) = \dfrac{1}{T_0}\displaystyle\sum_{n=-\infty}^{\infty} \left(A_m\tau \frac{\sin n\omega_0\tau/2}{n\omega_0\tau/2}\right) e^{jn\omega_0 t}$.

The frequency spectrum is shown in Figure 2.3.

Figure 2.3 Frequency spectrum of a periodic rectangular waveform.

Figure 2.4 shows the effect on the frequency spectrum of smaller τ [2].

Figure 2.4 Effect on frequency spectrum of smaller τ.

If the *bandwidth* B is specified as the width of the frequency band of a waveform from zero frequency to the first zero crossing, then $B = 1/\tau$ Hz.

If we let the pulse width τ in Figure 2.4 go to zero and the amplitude A_m go to infinity with $A_m\tau = 1$, all spectral lines in the frequency domain have unity length. Figure 2.5 shows the periodic unit impulses and the frequency spectrum of the periodic unit impulses [1]. The bandwidth becomes infinite.

Figure 2.5 (a) Periodic unit impulses, and (b) frequency spectrum.

Properties of the Complex Fourier Series [3]

1. If $s(t)$ is real,

$$c_n = c^*_{-n} \tag{2.11}$$

2. If $s(t)$ is real and even, $s(t) = s(-t)$,

$$\text{Img}[c_n] = 0 \tag{2.12}$$

3. If $s(t)$ is real and odd, $s(t) = - s(-t)$,

$$\text{Re}[c_n] = 0 \tag{2.13}$$

4. The complex Fourier-series coefficients of a real waveform are related to the quadrature Fourier-series coefficients by [1]

$$c_n = \begin{cases} a_n - jb_n, & n > 0 \\ a_0, & n = 0 \\ a_{-n} + jb_{-n}, & n < 0 \end{cases} \tag{2.14}$$

$$|c_n| = \sqrt{a_n^2 + b_n^2} \tag{2.15}$$

represents the *amplitude spectrum* and

$$\angle c_n = \theta_n = \tan^{-1} \frac{-b_n}{a_n} \tag{2.16}$$

represents the *phase spectrum* of the real waveform.

Proof.

$$s(t) = \frac{a_0}{T_0} + \frac{2}{T_0} \sum_{n=1}^{\infty} (a_n \cos n\omega_0 t + b_n \sin n\omega_0 t)$$

$$s(t) = \frac{a_0}{T_0} + \frac{1}{T_0} \sum_{n=1}^{\infty} [(a_n - jb_n)e^{j n\omega_0 t} + (a_n + jb_n)e^{-j n\omega_0 t}]$$

$$s(t) = \frac{a_0}{T_0} + \frac{1}{T_0}\sum_{n=1}^{\infty} [c_n e^{jn\omega_0 t} + c^*_n e^{-jn\omega_0 t}]$$

$$s(t) = \frac{a_0}{T_0} + \frac{1}{T_0}\sum_{n=1}^{\infty} [c_n e^{jn\omega_0 t} + c_{-n} e^{-jn\omega_0 t}]$$

$$s(t) = \frac{a_0}{T_0} + \left(\frac{1}{T_0}\sum_{n=1}^{\infty} c_n e^{jn\omega_0 t}\right) + \frac{1}{T_0}(c_{-1}e^{-j\omega_0 t} + c_{-2} e^{-j2\omega_0 t} + \ldots)$$

$$s(t) = \frac{a_0}{T_0} + \left(\frac{1}{T_0}\sum_{n=1}^{\infty} c_n e^{jn\omega_0 t}\right) + \left(\frac{1}{T_0}\sum_{n=-\infty}^{-1} c_n e^{jn\omega_0 t}\right)$$

$$s(t) = \frac{1}{T_0}\sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t}$$

Therefore, $|c_n| = \sqrt{a_n^2 + b_n^2}$. Also, we can write

$$s(t) = \frac{a_0}{T_0} + \frac{2\sqrt{a_n^2 + b_n^2}}{T_0}\sum_{n=1}^{\infty} \left(\frac{a_n}{\sqrt{a_n^2 + b_n^2}} \cos n\omega_0 t + \frac{b_n}{\sqrt{a_n^2 + b_n^2}} \sin n\omega_0 t\right)$$

$$s(t) = \frac{a_0}{T_0} + \frac{2}{T_0}\sum_{n=1}^{\infty} |c_n| \cos(n\omega_0 t + \theta_n)$$

where $\angle c_n = \theta_n = \tan^{-1} \frac{-b_n}{a_n}$. □

The equivalence between the Fourier series coefficients is demonstrated in Figure 2.6.

Figure 2.6 Fourier series coefficients, $n \geq 1$.

Parseval's Theorem for the Fourier Series [1, 4]

Parseval's Theorem for the Fourier series states that, if $s(t)$ is a periodic signal with period T_0, then the average normalised power (across a 1Ω resistor) of $s(t)$ is

$$P = \frac{1}{T_0} \int_{a - T_0/2}^{a + T_0/2} |s(t)|^2 \, dt = \sum_{n=-\infty}^{\infty} \left| \frac{c_n}{T_0} \right|^2 \tag{2.17}$$

If $s(t)$ is real, $|s(t)|$ is simply replaced by $s(t)$.

References

[1] M. Schwartz, Information Transmission, Modulation, and Noise, 4/e, McGraw-Hill, 1990.

[2] P. H. Young, Electronic Communication Techniques, 4/e, Prentice-Hall, 1998.

[3] L. W. Couch II, Digital and Analog Communication Systems, 5/e, Prentice Hall, 1997.

[4] H. P. Hsu, Analog and Digital Communications, McGraw-Hill, 1993.

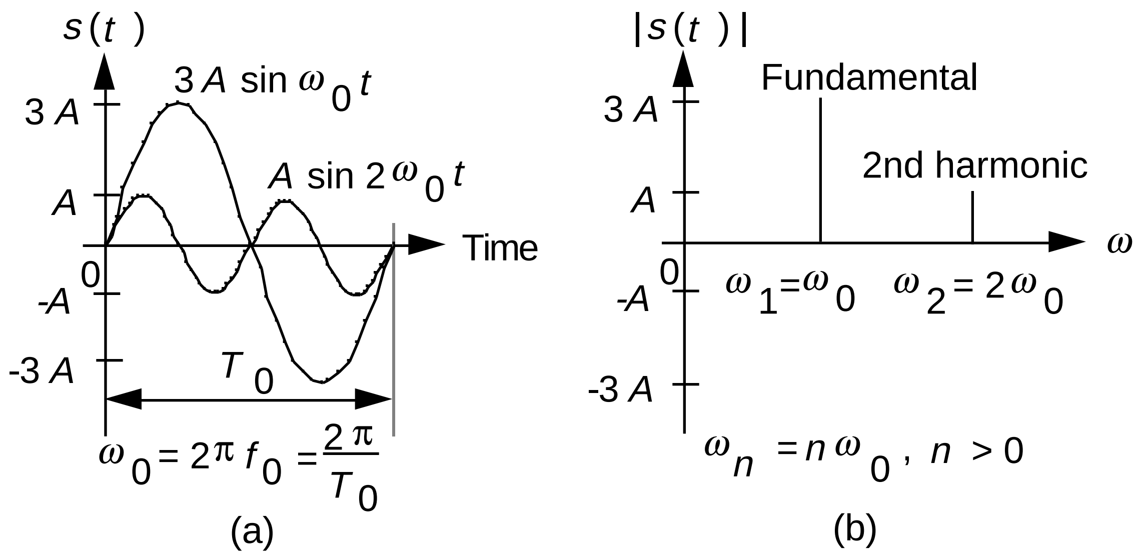

Figure 2.1 (a) Amplitude-time plot, and (b) amplitude-angular frequency plot.

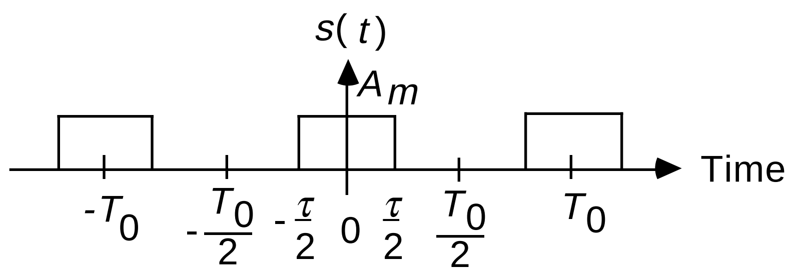

Figure 2.2 A periodic rectangular waveform.

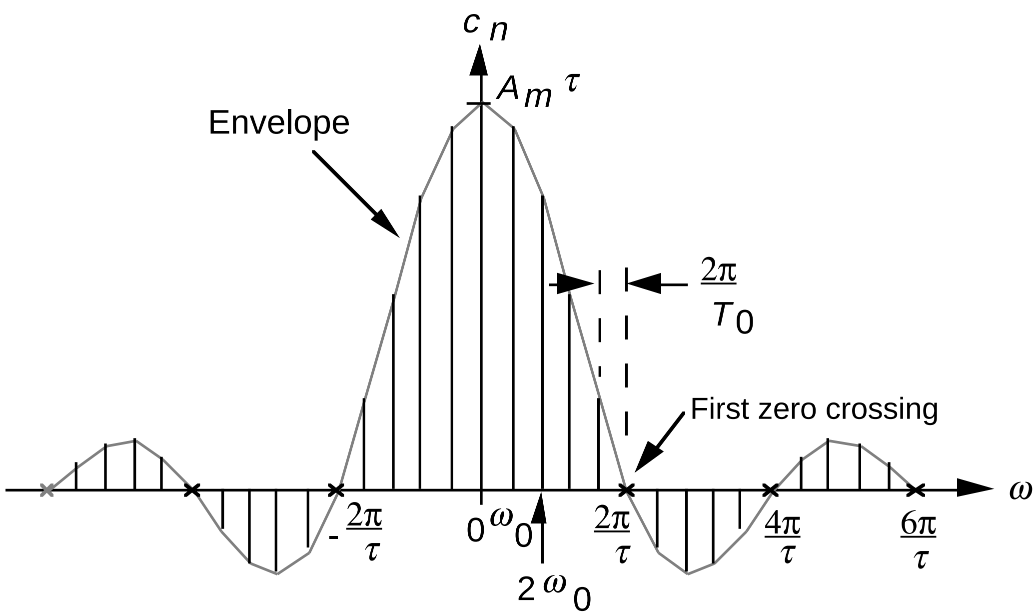

Figure 2.3 Frequency spectrum of a periodic rectangular waveform.

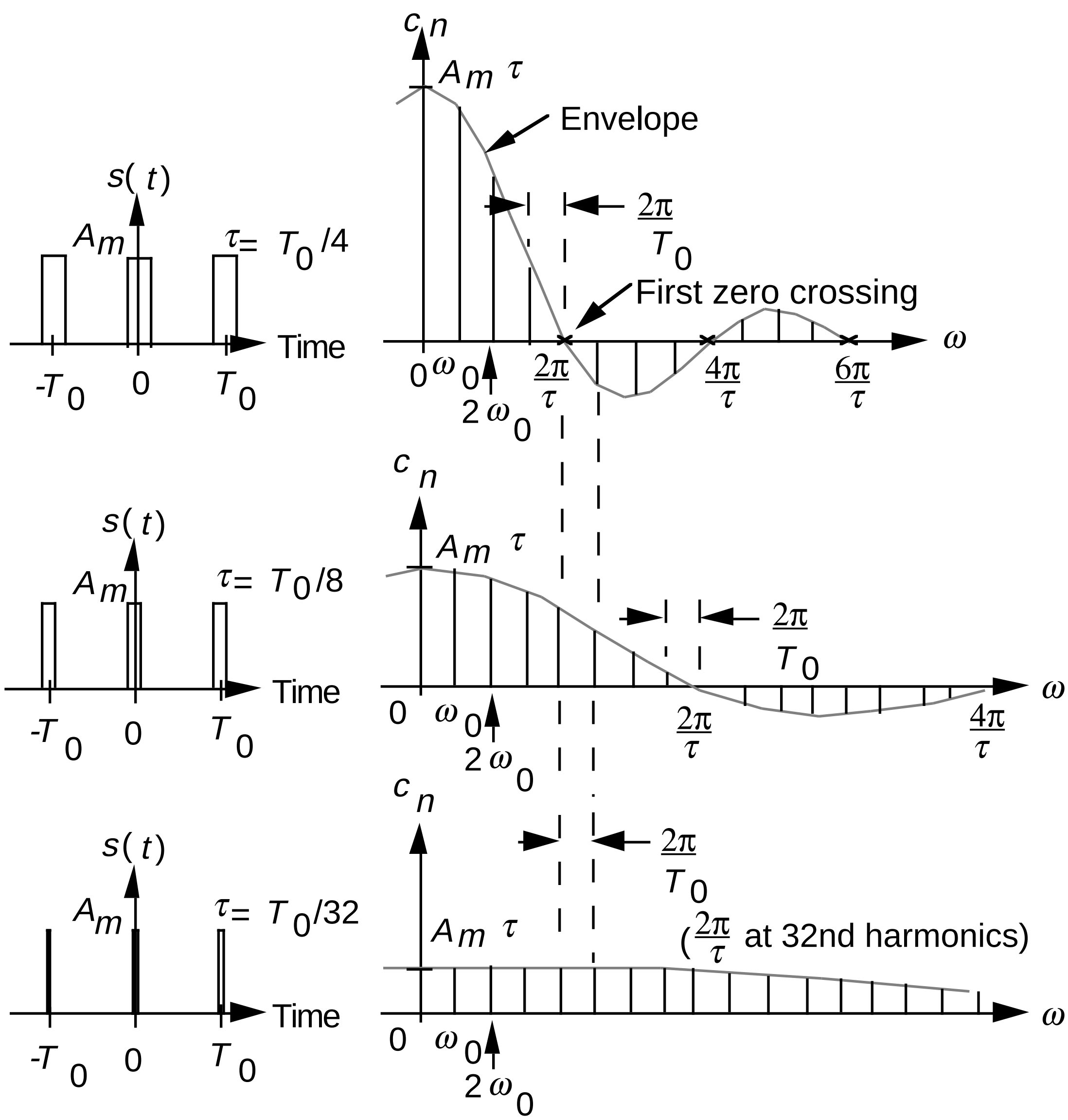

Figure 2.4 Effect on frequency spectrum of smaller τ.

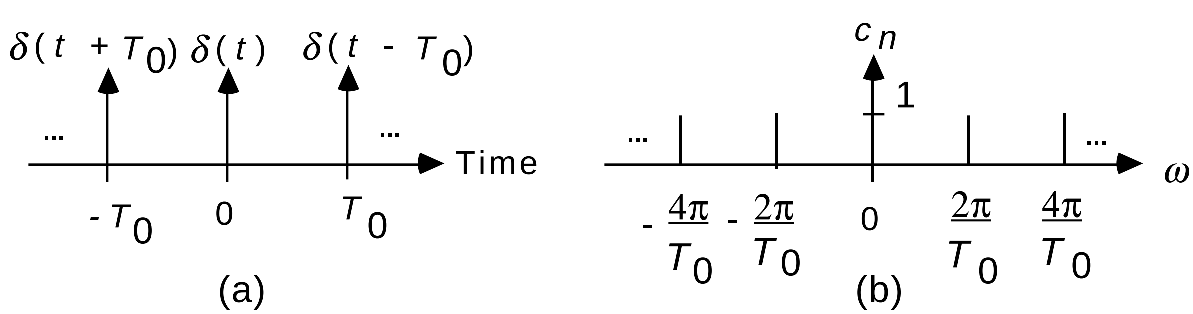

Figure 2.5 (a) Periodic unit impulses, and (b) frequency spectrum.

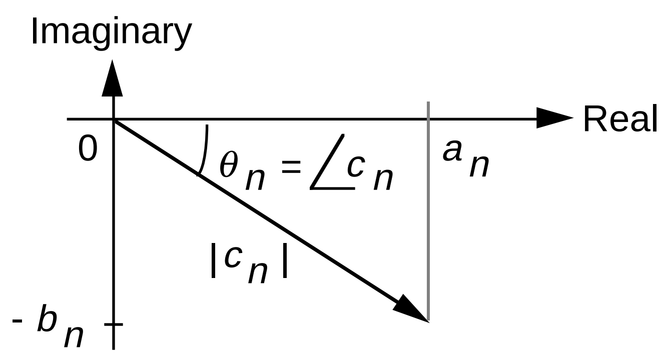

Figure 2.6 Fourier series coefficients, $n \geq 1$.