3. Fourier Transform

From Fourier Series to Fourier Transform [1, 2]

In communication systems, we often deal with non-periodic signals. An extension of the time-frequency relationship to a non-periodic signal $s(t)$ requires the introduction of the Fourier Integral. A nonperiodic signal can be viewed as a limiting case of a periodic signal, where the period T_0 approaches infinity. As T_0 approaches infinity, the periodic signal will eventually become a single non-periodic signal. This is shown in Figure 3.1.

Figure 3.1 Effect on frequency spectrum of increasing period T_0.

The normalised energy of the non-periodic signal becomes finite and its normalised power tends to zero.

Consider the amplitude spectrum of a periodic waveform as shown in Figure 3.2.

Figure 3.2 Amplitude spectrum of a periodic time function.

Let $\omega_n = n\omega_0$ and $\Delta\omega = \omega_{n+1} - \omega_n = \frac{2\pi}{T_0}$. The Fourier series of a periodic waveform $s(t)$ with period T_0 can be written as

$$s(t) = \frac{1}{T_0} \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t} = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t} \Delta\omega \tag{3.1}$$

and

$$c_n = \int_{a-T_0/2}^{a+T_0/2} s(t)\, e^{-jn\omega_0 t}\, dt \tag{3.2}$$

The constant a is usually set to 0. If T_0 approaches infinity, ω_0 goes to 0. The harmonics get closer and closer together. In the limit, the Fourier series summation representation of $s(t)$ becomes an integral, c_n becomes a continuous function $S(\omega)$, and we have a continuous frequency spectrum. In summary, as $T_0 \to \infty$, Σ becomes $\int$, ω_n becomes ω, and $\Delta\omega$ becomes $d\omega$. We have

$$s(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} c_n e^{j\omega t} d\omega \qquad (3.3)$$

and

$$S(\omega) = F[s(t)] = \lim_{T_0 \to \infty} c_n = \int_{-\infty}^{\infty} s(t)\, e^{-j\omega t}\, dt \qquad (3.4)$$

It is also very common to work in terms of frequency f, $f = \omega/2\pi$, because spectrum analysers are usually calibrated in hertz. Thus, we can express (3.3) and (3.4) as

$$s(t) = F^{-1}[S(f)] = \int_{-\infty}^{\infty} S(f)\, e^{j2\pi ft}\, df \qquad (3.5)$$

and

$$S(f) = F[s(t)] = \int_{-\infty}^{\infty} s(t)\, e^{-j2\pi ft}\, dt \qquad (3.6)$$

The functions $s(t)$ and $S(f)$ are said to constitute a Fourier transform pair, where $S(f)$ is the *Fourier transform* of a time function $s(t)$, and $s(t)$ is the *Inverse Fourier transform* (IFT) of a frequency-domain function $S(f)$.

Shorthand notation expressed in terms of t and f : $s(t)$ <-> $S(f)$
Shorthand notation expressed in terms of t and ω : $s(t)$ <-> $S(\omega)$

All physical waveforms encountered in engineering practice are Fourier transformable.

In general, $S(f)$ is a complex function of frequency.

In two-dimensional cartesian form, $S(f)$ can be expressed as

$$S(f) = X(f) + jY(f) \qquad (3.7)$$

In polar form, $S(f)$ can be expressed as

$$S(f) = |S(f)|\, e^{j\theta(f)} \qquad (3.8)$$

where

$$|S(f)| = \sqrt{X^2(f) + Y^2(f)} \text{ and } \theta(f) = \tan^{-1}\frac{Y(f)}{X(f)} \qquad (3.9)$$

$|S(f)|$ represents the *amplitude spectrum* and $\theta(f)$ represents the *phase spectrum* of $s(t)$.

Example 3.1 Find the spectrum of an exponential pulse $s(t) = \begin{cases} e^{-t}, & t > 0 \\ 0, & t < 0 \end{cases}$.

$$S(f) = \int_0^\infty e^{-t} e^{-j2\pi ft}\, dt$$

$$S(f) = \frac{1}{1+j\,2\pi f}$$

$$|S(f)| = \sqrt{\frac{1}{1+(2\pi f)^2}}$$

$$\theta(f) = -\tan^{-1}(2\pi f)$$

Transforms of Some Useful Functions [3]

1. Dirac Delta Time Function

$$F[\delta(t)] = \int_{-\infty}^{\infty} \delta(t)\, e^{-j2\pi ft}\, dt = 1$$

$\delta(t)$ <-> 1

Also, it can be shown that $\delta(t - t_0)$ <-> $e^{-j2\pi f t_0}$

2. Dirac Delta Frequency-Domain Function

$$F^{-1}[\delta(f)] = \int_{-\infty}^{\infty} \delta(f)\, e^{j2\pi ft}\, df = 1$$

1 <-> $\delta(f)$

Also, it can be shown that $e^{j2\pi f_0 t}$ <-> $\delta(f - f_0)$

Example 3.2 Find the spectrum of a sinusoid $v(t) = A \sin 2\pi f_0 t = A(\frac{e^{j2\pi f_0 t} - e^{-j2\pi f_0 t}}{2j})$.

Since $e^{j2\pi f_0 t}$ <-> $\delta(f - f_0)$, we have

$$V(f) = \frac{A}{2j}\delta(f - f_0) - \frac{A}{2j}\delta(f + f_0)$$
$$V(f) = -\frac{A}{2}j[\delta(f - f_0) - \delta(f + f_0)]$$

Figure 3.3 Spectrum of the periodic function $v(t) = A \sin 2\pi f_0 t$.

3. Rectangular, sin *x*/*x*, and Triangular Pulses

Figure 3.4 Spectra of (a) rectangular, (b) sin *x*/*x*, and (c) triangular pulses.

Observations:

1. Figure 3.4a - Spectrum spreads out as the pulse width T decreases. Bandwidth $B = 1/T$ Hz and $S(f)$ decreases as $1/f$.

2. Figure 3.4c - Spectum spreads out as the pulse width T decreases. Bandwidth $B = 1/T$ Hz and $S(f)$ decreases as $1/f^2$.

The smoother the time-domain function, the more rapidly the spectrum decreases with increasing frequency, packing more frequency contents into a specified bandwidth.

An inverse time-bandwidth relation always exists.

Bandwidth plays a significant role in determining *transmission rate*.

Function	$s(t)$	$S(f)$
Rectangular	$\Pi\left(\frac{t}{T}\right)$	$T\left[\frac{\sin \pi f T}{\pi f T}\right]$
Triangular	$\Lambda\left(\frac{t}{T}\right)$	$T\left[\frac{\sin \pi f T}{\pi f T}\right]^2$
Unit step	$u(t) = \begin{cases} 1, & t > 0 \\ 0, & t < 0 \end{cases}$	$\frac{1}{2}\delta(f) + \frac{1}{j\ 2\pi f}$
Signum	$\text{sgn}\ (t) = \begin{cases} 1, & t > 0 \\ -1, & t < 0 \end{cases}$	$\frac{1}{j\pi f}$
Constant	1	$\delta(f)$
Impulse at $t = t_0$	$\delta(t - t_0)$	$e^{-j\ 2\pi f t_0}$
Sinc	$\frac{\sin\ 2\pi\ B\ t}{2\pi\ B\ t}$	$\frac{1}{2B}\ \Pi\left(\frac{f}{2B}\right)$, B denotes bandwidth
Phasor	$e^{j\ (2\pi f_0 t + \varphi)}$	$e^{j\ \varphi}\delta(f - f_0)$
Cosine	$\cos(2\pi f_c t + \varphi)$	$\frac{1}{2}[e^{j\ \varphi}\delta(f - f_c) + e^{-j\varphi}\delta(f + f_c)]$
Sine	$\sin(2\pi f_c t + \varphi)$	$\frac{1}{2j}[e^{j\ \varphi}\delta(f - f_c) - e^{-j\varphi}\delta(f + f_c)]$
Exponential, one – sided	$\begin{cases} e^{-t/T}, & t > 0 \\ 0, & t < 0 \end{cases}$	$\frac{T}{1 + j\ 2\pi\ f T}$
Exponential, two – sided	$e^{-\|t\|/T}$	$\frac{2T}{1 + (\ 2\pi\ f T\ )^2}$
Impulse train	$\sum_{k=-\infty}^{\infty} \delta(t - kT_0)$	$f_0 \sum_{n=-\infty}^{\infty} \delta(f - nf_0)$, where $f_0 = 1/T_0$

Table 3.1a Some Fourier transform pairs expressed in terms of t and f.

Function	$s(t)$	$S(\omega)$
Rectangular	$\Pi\left(\frac{t}{T}\right)$	$T\,\frac{\sin\ \omega T\ /\ 2}{\omega T\ /2}$
Triangular	$\Lambda\left(\frac{t}{T}\right)$	$T[\,\frac{\sin\ \omega T\ /\ 2}{\omega T\ /2}\,]^2$
Unit step	$u(t) = \begin{cases} 1, & t > 0 \\ 0, & t < 0 \end{cases}$	$\pi\delta(\omega) + \frac{1}{j\omega}$
Signum	$\mathrm{sgn}\ (t)= \begin{cases} 1, & t > 0 \\ -1, & t < 0 \end{cases}$	$\frac{2}{j\omega}$
Constant	1	$2\pi\delta(\omega)$
Impulse at $t= t_0$	$\delta(t - t_0)$	$e^{-j\ \omega t_0}$
Sinc	$\frac{\sin\ 2\pi\ B\,t}{2\,\pi\ B\ t}$	$\frac{1}{2B}\ \Pi\left(\frac{\omega}{4\pi\ B}\right)$, B denotes bandwidth
Phasor	$e^{j\ (\omega_0 t+\varphi)}$	$2\pi\ e^{j\ \varphi}\delta(\omega - \omega_0)$
Cosine	$\cos(\omega_c t+\varphi)$	$\pi[e^{j\ \varphi}\delta(\omega - \omega_c) + e^{-j\varphi}\delta(\omega + \omega_c)]$
Sine	$\sin(\omega_c t+\varphi)$	$\frac{\pi}{j}\ [e^{j\ \varphi}\delta(\omega - \omega_c) - e^{-j\varphi}\delta(\omega + \omega_c)]$
Exponential, one – sided	$\begin{cases} e^{-t\,/\,T}, & t > 0 \\ 0, & t < 0 \end{cases}$	$\frac{T}{1+j\omega T}$
Exponential, two – sided	$e^{-\lvert t\rvert/T}$	$\frac{2T}{1+(\omega T\ )^2}$
Impulse train	$\sum_{k\ =-\infty}^{\infty}\ \delta(t- kT_0)$	$\omega_0 \sum_{n\ =-\infty}^{\infty}\ \delta(\omega - n\ \omega_0)$, where $\omega_0 = 2\pi/T_0$

Table 3.1b Some Fourier transform pairs expressed in terms of t and ω.

Properties of Fourier Transforms [3-5]

1. *Symmetry (Duality) Property*

$S(t)$ <-> $s(-f)$

Proof.

$$s(t) = \int_{-\infty}^{\infty} S(f)\, e^{j2\pi ft}\, df$$
$$s(t) = \int_{-\infty}^{\infty} S(x)\, e^{j2\pi xt}\, dx$$
$$s(-t) = \int_{-\infty}^{\infty} S(x)\, e^{-j2\pi xt}\, dx$$
$$s(-f) = \int_{-\infty}^{\infty} S(x)\, e^{-j2\pi xf}\, dx$$
$$s(-f) = \int_{-\infty}^{\infty} S(t)\, e^{-j2\pi tf}\, dt$$

Hence, $S(t)$ <-> $s(-f)$. □

2. *Scaling Property*

$$s(at) \;\text{<->}\; \frac{1}{|a|}\, S\left(\frac{f}{a}\right)$$

Proof.

$$F[s(at)] = \int_{-\infty}^{\infty} s(at)\, e^{-j2\pi ft}\, dt$$

Let $t_1 = at$ and $a > 0$, we get

$$F[s(t_1)] = (1/a) \int_{-\infty}^{\infty} s(t_1)\, e^{-j2\pi(f/a)t_1}\, dt_1$$
$$F[s(t_1)] = (1/a)\, S(f/a)$$

For $a < 0$, we get

$$F[s(t_1)] = (-1/a) \int_{-\infty}^{\infty} s(t_1)\, e^{-j2\pi(f/a)t_1}\, dt_1$$

$$F[s(t_1)] = (-1/a)\, S(f/a)$$

Hence, $s(at) \text{ <-> } \frac{1}{|a|} S(\frac{f}{a})$. □

3. *Time Shifting (Time Delay) Property*

$$s(t - T_d) \text{ <-> } S(f)e^{-j\,2\pi f T_d}$$

Proof.

Let $\alpha = t - T_d$, $d\alpha = dt$.

$$F[s(t - T_d)] = \int_{-\infty}^{\infty} s(t - T_d)\, e^{-j\,2\pi f t}\, dt$$

$$= \int_{-\infty}^{\infty} s(\alpha)\, e^{-j\,2\pi f(\alpha + T_d)} d\alpha$$

$$= e^{-j\,2\pi f T_d} \int_{-\infty}^{\infty} s(\alpha)\, e^{-j\,2\pi f \alpha} d\alpha$$

$$= e^{-j\,2\pi f T_d}\, S(f)$$

Hence, $s(t - T_d) \text{ <-> } S(f)e^{-j\,2\pi f T_d}$. □

4. *Frequency Shifting Property*

$$s(t)e^{j2\pi f_c t} \text{ <-> } S(f-f_c)$$

Proof.

$$S(f) = \int_{-\infty}^{\infty} s(t)\, e^{-j2\pi f t}\, dt$$

$$S(f - f_c) = \int_{-\infty}^{\infty} s(t)\, e^{-j\,2\pi(f-f_c)t}\, dt$$

$$= \int_{-\infty}^{\infty} [s(t)\;\; e^{j\,2\pi f_c t}]\, e^{-j2\pi f t}\, dt$$

Hence, $s(t)e^{j2\pi f_c t}$ <-> $S(f-f_c)$. □

5. *Differentiation Property*

$$\frac{d^n s(t)}{dt^n} \leftrightarrow (j2\pi f)^n S(f)$$

Proof.

Direct differentation of the inverse Fourier transform $s(t) = F^{-1}[S(f)] = \int_{-\infty}^{\infty} S(f)\, e^{j2\pi ft}\, df$ with respect to time n times.

$$ds / dt = \int_{-\infty}^{\infty} j2\pi f\, S(f)\, e^{j2\pi ft}\, df$$

$$ds / dt \leftrightarrow j2\pi f\, S(f)$$

$$d^2s / dt^2 = \int_{-\infty}^{\infty} (j2\pi f)^2\, S(f)\, e^{j2\pi ft}\, df$$

$$d^2s / dt^2 \leftrightarrow (j2\pi f)^2\, S(f)$$

:

$$\frac{d^n s(t)}{dt^n} = \int_{-\infty}^{\infty} (j2\pi f)^n S(f)\, e^{j2\pi ft}\, df$$

Hence, $\frac{d^n s(t)}{dt^n} \leftrightarrow (j2\pi f)^n S(f)$. □

Differentiation increases the high-frequency content of a signal. The derivative of an even function must be odd. Hence, the Fourier transform of the derivative of the function must be odd and imaginary.

6. *Convolution Property*

$$s_1(t) * s_2(t) \leftrightarrow S_1(f)S_2(f)$$

Proof.

$$F[s_1(t) * s_2(t)] = \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} s_1(\lambda)\, s_2(t - \lambda)\, d\lambda \right] e^{-j2\pi ft}\, dt$$

$$F[s_1(t) * s_2(t)] = \int_{-\infty}^{\infty} s_1(\lambda)[\int_{-\infty}^{\infty} s_2(t - \lambda)\, e^{-j2\pi ft}\, dt]\, d\lambda$$

Since $s(t - T_d) \text{<->} S(f)\, e^{-j\,2\pi f T_d}$ (time shifting property), the inner integral is the Fourier transform of $s_2(t - \lambda)$. We can write

$$F[s_1(t) * s_2(t)] = \int_{-\infty}^{\infty} s_1(\lambda) S_2(f) e^{-j2\pi f\lambda}\, d\lambda$$

$$F[s_1(t) * s_2(t)] = S_2(f) \int_{-\infty}^{\infty} s_1(\lambda) e^{-j2\pi f\lambda}\, d\lambda$$

$$F[s_1(t) * s_2(t)] = S_1(f) S_2(f)$$

Therefore, $s_1(t) * s_2(t) \text{<->} S_1(f)S_2(f)$ □

7. *Integration Property* [4]

$$\int_{-\infty}^{t} s(\lambda)\, d\lambda \text{ <-> } \frac{1}{j\ 2\pi f} S(f) + \frac{1}{2} S(0)\ \delta(f)$$

Proof.

Because $u(t - \lambda) = \begin{cases} 1, & \lambda \le t \\ 0, & \lambda > t \end{cases}$, it follows that

$$s(t) * u(t) = \int_{-\infty}^{\infty} s(\lambda)\, u(t-\lambda)\, d\lambda = \int_{-\infty}^{t} s(\lambda)\, d\lambda$$

where $u(t)$ is a unit step function and the Fourier transform of $u(t)$ is $U(f) = \frac{1}{2}\delta(f) + \frac{1}{j\ 2\pi f}$. It follows from the time convolution property that $s(t) * u(t) \text{ <-> } S(f)U(f)$ and $S(f)U(f) = S(f)[\frac{1}{2}\delta(f) + \frac{1}{j\ 2\pi f}] = \frac{1}{j\ 2\pi f} S(f) + \frac{1}{2} S(0)\delta(f)$. Hence we have

$$\int_{-\infty}^{t} s(\lambda)\, d\lambda \text{ <-> } \frac{1}{j\ 2\pi f} S(f) + \frac{1}{2} S(0)\ \delta(f).$$ □

Operation	Function	Fourier Transform
Linearity	$a_1 s_1(t) + a_2 s_2(t)$	$a_1 S_1(f) + a_2 S_2(f)$
Conjugation	$s^*(t)$	$S^*(-f)$
Symmetry	$S(t)$	$s(-f)$
Scaling	$s(at)$	$\frac{1}{\lvert a \rvert} S(\frac{f}{a})$
Time reversal	$s(-t)$	$S(-f)$
Time shift (delay)	$s(t - T_d)$	$S(f)\, e^{-j\, 2\pi f T_d}$
Frequency shift	$s(t) e^{j2\pi f_c t}$	$S(f-f_c)$
Real signal frequency translation	$s(t)\ \cos(2\pi f_c t + \theta)$	$\frac{1}{2}[e^{j\theta} S(f-f_c) + e^{-j\theta} S(f+f_c)]$
Bandpass signal	$\mathrm{Re}\{g(t) e^{j2\pi f_c t}\}$	$\frac{1}{2}[G(f-f_c) + G^*(-f-f_c)]$
Differentiation	$\frac{d^n s(t)}{dt^n}$	$(j2\pi f)^n\, S(f)$
Integration	$\int_{-\infty}^{t} s(\lambda)\, d\lambda$	$\frac{1}{j\ 2\pi\ f} S(f) + \frac{1}{2} S(0)\ \delta(f)$
Convolution	$s_1(t) * s_2(t) = \int_{-\infty}^{\infty} s_1(\lambda)\, s_2(t-\lambda)\, d\lambda$	$S_1(f) S_2(f)$
Multiplication	$s_1(t) s_2(t)$	$S_1(f) * S_2(f) = \int_{-\infty}^{\infty} S_1(\lambda)\, S_2(f-\lambda)\, d\lambda$

Table 3.2a Some Fourier transform properties expressed in terms of *t* and *f*.

Operation	Function	Fourier Transform
Linearity	$a_1 s_1(t) + a_2 s_2(t)$	$a_1 S_1(\omega) + a_2 S_2(\omega)$
Conjugation	$s^*(t)$	$S^*(-\omega)$
Symmetry	$S(t)$	$2\pi s(-\omega)$
Scaling	$s(at)$	$\frac{1}{\lvert a \rvert} S(\frac{\omega}{a})$
Time reversal	$s(-t)$	$S(-\omega)$
Time shift (delay)	$s(t - T_d)$	$S(\omega)\, e^{-j\omega T_d}$
Frequency shift	$s(t)e^{j\omega_c t}$	$S(\omega-\omega_c)$
Real signal frequency translation	$s(t)\cos(\omega_c t + \theta)$	$\frac{1}{2}[e^{j\theta}S(\omega-\omega_c)+e^{-j\theta}S(\omega+\omega_c)]$
Bandpass signal	$\mathrm{Re}\{g(t)e^{j\omega_c t}\}$	$\frac{1}{2}[G(\omega - \omega_c) + G^*(-\omega - \omega_c)]$
Differentiation	$\frac{d^n s(t)}{dt^n}$	$(j\omega)^n S(\omega)$
Integration	$\int_{-\infty}^{t} s(\lambda)\, d\lambda$	$\frac{1}{j\omega} S(\omega) + \pi S(0)\, \delta(\omega)$
Convolution	$s_1(t) * s_2(t) = \int_{-\infty}^{\infty} s_1(\lambda)\, s_2(t-\lambda)\, d\lambda$	$S_1(\omega) S_2(\omega)$
Multiplication	$s_1(t) s_2(t)$	$\frac{1}{2\pi} S_1(\omega) * S_2(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_1(\lambda)\, S_2(\omega-\lambda)\, d\lambda$

Table 3.2b Some Fourier transform properties expressed in terms of t and ω.

Example 3.3 Use the scaling and real-signal frequency-translation properties to find the Fourier transform of a damped sinusoid $s(t) = \begin{cases} e^{-t/T} \sin \omega_0 t, & t > 0, T > 0 \\ 0, & t < 0 \end{cases}$.

From Example 3.1 we have

$$\begin{Bmatrix} e^{-t}, & t > 0 \\ 0, & t < 0 \end{Bmatrix} <-> \frac{1}{1+j\ 2\pi f}$$

Using the scaling property with $a = 1/T$, we get

$$\begin{Bmatrix} e^{-t/T}, & t > 0 \\ 0, & t < 0 \end{Bmatrix} <-> \frac{T}{1+j\ 2\pi\ fT}$$

Using the real-signal frequency-translation property with $\theta = -\pi/2$, we get

$$S(f) = \frac{1}{2}\,[e^{-j\pi/2}\frac{T}{1+j2\pi(f-f_0)T} + e^{j\pi/2}\frac{T}{1+j2\pi(f+f_0)T}]$$

The sin $\omega_0 t$ factor causes the spectrum to move from $f = 0$ to $f = \pm f_0$.

8. If $s(t)$ is real, then

$$S(-f) = S^*(f) \tag{3.10}$$

Proof. $S(-f) = \int_{-\infty}^{\infty} s(t)\ e^{j2\pi ft}\ dt$ and $S^*(f) = \mathcal{F}[s(t)]^* = [\int_{-\infty}^{\infty} s(t)\ e^{-j2\pi ft}\ dt]^* = \int_{-\infty}^{\infty} s^*(t)\ e^{j2\pi ft}\ dt$. Because $s(t)$ is real, $s^*(t) = s(t)$ and $S(-f) = S^*(f)$. □

9. If $s(t)$ is real, then

$$|S(-f)| = |S(f)| \tag{3.11}$$

and

$$\theta(-f) = -\theta(f) \tag{3.12}$$

Proof. $S(-f) = |S(-f)|\ e^{j\theta(-f)}$ and $S^*(f) = |S(f)|\ e^{-j\theta(f)}$. Because $s(t)$ is real, $S(-f) = S^*(f)$ and we see that (3.11) and (3.12) are true. □

$S(f)$ can be complex even though $s(t)$ is real.

If $s(t)$ is a:	Then $S(f)$ is a:
Real and even function of t	Real and even function of f
Real and odd	Imaginary and odd
Imaginary and even	Imaginary and even
Imaginary and odd	Real and odd
Complex and even	Complex and even
Complex and odd	Complex and odd

Table 3.3 Fourier transform properties for various forms of $s(t)$ [2].

Example 3.4

$$\cos 2\pi f_c t = \frac{e^{j\,2\pi f_c t} + e^{-j\,2\pi f_c t}}{2} \quad \text{<->} \quad \frac{1}{2}\,[\delta(f-f_c) + \delta(f+f_c)]$$

$$\sin 2\pi f_c t = \frac{e^{j\,2\pi f_c t} - e^{-j\,2\pi f_c t}}{2j} \quad \text{<->} \quad \frac{1}{2j}\,[\delta(f-f_c) - \delta(f+f_c)]$$

Figure 3.5 Fourier transform spectrum of (a) $\cos 2\pi f_c t$, and (b) $\sin 2\pi f_c t$.

Observations:

1. Figure 3.5a - A real and even function in t gives a real and even function in f.
2. Figure 3.5b - A real and odd function in t gives an imaginary and odd function in f.

Parseval's Theorem for the Fourier Transform and Energy Spectral Density [4, 5]

Parseval's Theorem for the Fourier transform states that if $s_1(t)$ and $s_2(t)$ are two complex energy signals, then

$$\int_{-\infty}^{\infty} s_1(t)\, s^*_2(t)\, dt = \int_{-\infty}^{\infty} S_1(f)\, S^*_2(f)\, df \tag{3.13}$$

Proof.

$$\int_{-\infty}^{\infty} s_1(t)\, s^*_2(t)\, dt = \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} S_1(f)\, e^{j2\pi ft}\, df \right] s^*_2(t)\, dt$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} S_1(f)\, s^*_2(t)\, e^{j2\pi ft}\, df\, dt$$

Interchanging the order of integration, we have

$$\int_{-\infty}^{\infty} s_1(t)\, s^*_2(t)\, dt = \int_{-\infty}^{\infty} S_1(f) \left[\int_{-\infty}^{\infty} s_2(t) e^{-j2\pi ft}\, dt \right]^* df$$

$$= \int_{-\infty}^{\infty} S_1(f) [F[s_2(t)]]^*\, df$$

$$= \int_{-\infty}^{\infty} S_1(f)\, S^*_2(f)\, df \qquad \square$$

If $s_1(t) = s_2(t)$, then *Rayleigh's energy theorem* states that the normalised energy is

$$E = \int_{-\infty}^{\infty} |s_1(t)|^2\, dt = \int_{-\infty}^{\infty} |S_1(f)|^2\, df \tag{3.14}$$

The *energy spectral density* (*ESD*) is defined for energy waveforms by

$$E_{11}(f) = |S_1(f)|^2 \tag{3.15}$$

Proof.

The auto-correlation of a complex energy waveform $s_1(t)$ is

$$R_{11}(\tau) = \int_{-\infty}^{\infty} s^*_1(t)\, s_1(t+\tau)\, dt$$

$$R_{11}(0) = \int_{-\infty}^{\infty} s^*_1(t)\, s_1(t)\, dt$$

$$= \int_{-\infty}^{\infty} |s_1(t)|^2\, dt$$

$$= E = \int_{-\infty}^{\infty} |S_1(f)|^2\, df$$

and the Fourier transform of $R_{11}(\tau)$ is

$$E_{11}(f) = \mathcal{F}[R_{11}(\tau)] = \int_{-\infty}^{\infty} R_{11}(\tau)\, e^{-j2\pi f\tau}\, d\tau$$

$$R_{11}(\tau) = \mathcal{F}^{-1}[E_{11}(f)] = \int_{-\infty}^{\infty} E_{11}(f)\, e^{j2\pi f\tau}\, df$$

$$R_{11}(0) = \int_{-\infty}^{\infty} E_{11}(f) df$$

Hence $E = \int_{-\infty}^{\infty} E_{11}(f) df$ and $E_{11}(f) = |S_1(f)|^2$. □

Power Spectral Density and Wiener-Khintchine Theorem [3-5]

The power spectral density and related concepts for a power waveform $s_2(t)$ can be readily understood by defining a truncated waveform $s_1(t)$ as

$$s_1(t) = \begin{cases} s_2(t), & -T/2 < t < T/2 \\ 0, & elsewhere \end{cases} = s_2(t)\Pi\left(\frac{t}{T}\right) \tag{3.16}$$

and $s_1(t)$ is an energy waveform as long as T is finite.

Let $P_{22}(f)$ be the *power spectral density* of a power waveform $s_2(t)$. The *Wiener-Khintchine theorem* states that the power spectral density and the autocorrelation function are Fourier transform pairs.

$$R_{22}(\tau) \leftrightarrow P_{22}(f) \tag{3.17}$$

Furthermore, the average normalised power is

$$P = \langle s_2^2(t) \rangle = S_{2rms}^2 = \int_{-\infty}^{\infty} P_{22}(f)\, df = R_{22}(0)$$

Proof.

The average normalised power of a complex power waveform $s_2(t)$ is

$$P = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} s_2(t) s^*_2(t) dt$$

$$P = \lim_{T \to \infty} \frac{1}{T} \int_{-\infty}^{\infty} s_1(t) s^*_1(t) dt = \lim_{T \to \infty} \frac{E}{T}$$

where E is the normalised energy of the truncated waveform $s_1(t)$.

The auto-correlation of a complex power waveform $s_2(t)$ is

$$R_{22}(\tau) = \langle s^*_2(t)\, s_2(t+\tau) \rangle = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} s^*_2(t)\, s_2(t+\tau) dt$$

$$\begin{aligned} R_{22}(0) &= \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} s^*_2(t)\, s_2(t)\, dt \\ &= \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} |s_2(t)|^2\, dt \\ &= P \end{aligned}$$

and the Fourier transform of $R_{22}(\tau)$ is

$$P_{22}(f) = \mathcal{F}[R_{22}(\tau)] = \int_{-\infty}^{\infty} R_{22}(\tau)\, e^{-j2\pi f\tau}\, d\tau$$

$$R_{22}(\tau) = \mathcal{F}^{-1}[P_{22}(f)] = \int_{-\infty}^{\infty} P_{22}(f)\, e^{j2\pi f\tau}\, df$$

$$R_{22}(0) = \int_{-\infty}^{\infty} P_{22}(f) df$$

Hence $P = \int_{-\infty}^{\infty} P_{22}(f) df$. □

The average normalised power of a power waveform is now related to the power spectral density.

The *power spectral density* (*PSD*) for a power waveform $s_2(t)$ is

$$P_{22}(f) = \lim_{T \to \infty} \left(\frac{|S_1(f)|^2}{T} \right) \qquad (3.18)$$

where $S_1(f)$ is the Fourier transform of the truncated waveform $s_1(t)$.

Proof.

$$R_{11}(\tau) \text{ <-> } E_{11}(f) = |S_1(f)|^2$$

and

$$R_{22}(\tau) \text{ <-> } P_{22}(f)$$

The auto-correlation of a complex power waveform $s^*_2(t)$ is

$$R_{22}(\tau) = \langle s^*_2(t)\, s_2(t+\tau)\rangle = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} s^*_2(t)\, s_2(t+\tau)dt$$

$$R_{22}(\tau) = \lim_{T\to\infty} \frac{1}{T}\int_{-\infty}^{\infty} s^*_1(t)\, s_1(t+\tau)dt = \lim_{T\to\infty} \frac{R_{11}(\tau)}{T}$$

Hence we have

$$P_{22}(f) = \lim_{T\to\infty} \frac{E_{11}(f)}{T} = \lim_{T\to\infty}\left(\frac{|S_1(f)|^2}{T}\right). \qquad \square$$

The power spectral density is always a real nonnegative function of frequency. It is not sensitive to the phase spectrum of the truncated waveform $s_1(t)$. Thus, $A \sin 2\pi f_0 t$ and $A \cos 2\pi f_0 t$ have the same PSD because the phase has no effect on the power spectral density.

Energy Signal	Power Signal
$E = \int_{-\infty}^{\infty} s_1(t)\, s^*_1(t)\, dt$	$P = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} s_2(t) s^*_2(t) dt$
$= \int_{-\infty}^{\infty} S_1(f)\, S^*_1(f)\, df$	$= \lim_{T \to \infty} \frac{E}{T}$
$R_{11}(\tau) = \int_{-\infty}^{\infty} s^*_1(t) s_1(t+\tau) dt$	$R_{22}(\tau) = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} s^*_2(t) s_2(t+\tau) dt$ $= \lim_{T \to \infty} \frac{R_{11}(\tau)}{T}$
$E_{11}(f) = \lvert S_1(f)\rvert^2$	$P_{22}(f) = \lim_{T \to \infty} \frac{E_{11}(f)}{T} = \lim_{T \to \infty} \left(\frac{\lvert S_1(f)\rvert^2}{T} \right)$
$R_{11}(\tau) \leftrightarrow E_{11}(f)$	$R_{22}(\tau) \leftrightarrow P_{22}(f)$
$E = \int_{-\infty}^{\infty} E_{11}(f)\, df$	$P = \int_{-\infty}^{\infty} P_{22}(f)\, df$

Table 3.4 Relationships for energy and power signals.

Fourier Transform of Periodic Signals [1]

So far we have used the Fourier series and the Fourier transform to represent periodic and nonperiodic signals, respectively. For periodic signals, we can use an impulse function in the frequency domain to represent discrete components of periodic signals using Fourier transforms. With this approach, both periodic and nonperiodic signals can be incorporated in a common Fourier-transform framework.

Recall:

$$A\delta(t) \leftrightarrow A$$
$$A\delta(t - t_0) \leftrightarrow Ae^{-j2\pi f t_0}$$
$$A \leftrightarrow A\delta(f)$$
$$Ae^{j2\pi f_0 t} \leftrightarrow A\delta(f - f_0)$$

The complex Fourier series of a periodic signal is given by

$$s(t) = \frac{1}{T_0} \sum_{n=-\infty}^{\infty} c_n e^{j2\pi n f_0 t}$$

and the Fourier transform of $s(t)$ is

$$S(f) = \frac{1}{T_0} \sum_{n=-\infty}^{\infty} c_n \delta(f - nf_0)$$

Example 3.5 The complex Fourier series of a periodic rectangular waveform $s(t)$ is

$$s(t) = \frac{1}{T_0} \sum_{n=-\infty}^{\infty} c_n e^{j2\pi n f_0 t} <-> \frac{1}{T_0} \sum_{n=-\infty}^{\infty} c_n \delta(f - nf_0)$$

where

$$c_n = A_m \tau \left(\frac{\sin 2\pi n f_0 \tau / 2}{2\pi n f_0 \tau / 2} \right).$$

Figure 3.6 (a) A periodic rectangular waveform $s(t)$, and (b) the Fourier transform spectrum of $s(t)$.

Example 3.6 A periodic impulse $s(t)$ is

$$s(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT_0) <-> f_0 \sum_{n=-\infty}^{\infty} \delta(f - nf_0), \text{ where } f_0 = 1/T_0.$$

Figure 3.7 (a) Periodic impulse $s(t)$, and (b) Fourier transform spectrum of $s(t)$.

Summary

For periodic signals, the impulse function $\delta(f)$ provides a unified method of describing such signals in the frequency domain using the Fourier transform.

1. Express the periodic signal in a complex Fourier series.
2. Take the Fourier transform.

If a signal $s(t) = s_p(t)s_a(t)$, where $s_p(t)$ and $s_a(t)$ are the corresponding periodic and nonperiodic components, respectively, we have

$$s(t) = s_p(t)s_a(t) \leftrightarrow S_p(f) * S_a(f)$$

If a signal $s(t) = s_p(t) + s_a(t)$, we have

$$s(t) = s_p(t) + s_a(t) \leftrightarrow S_p(f) + S_a(f)$$

In using the impulse function in the frequency domain, we must bear in mind that we are dealing with signals having infinite or undefined energy, and the concept of energy spectral density no longer exists.

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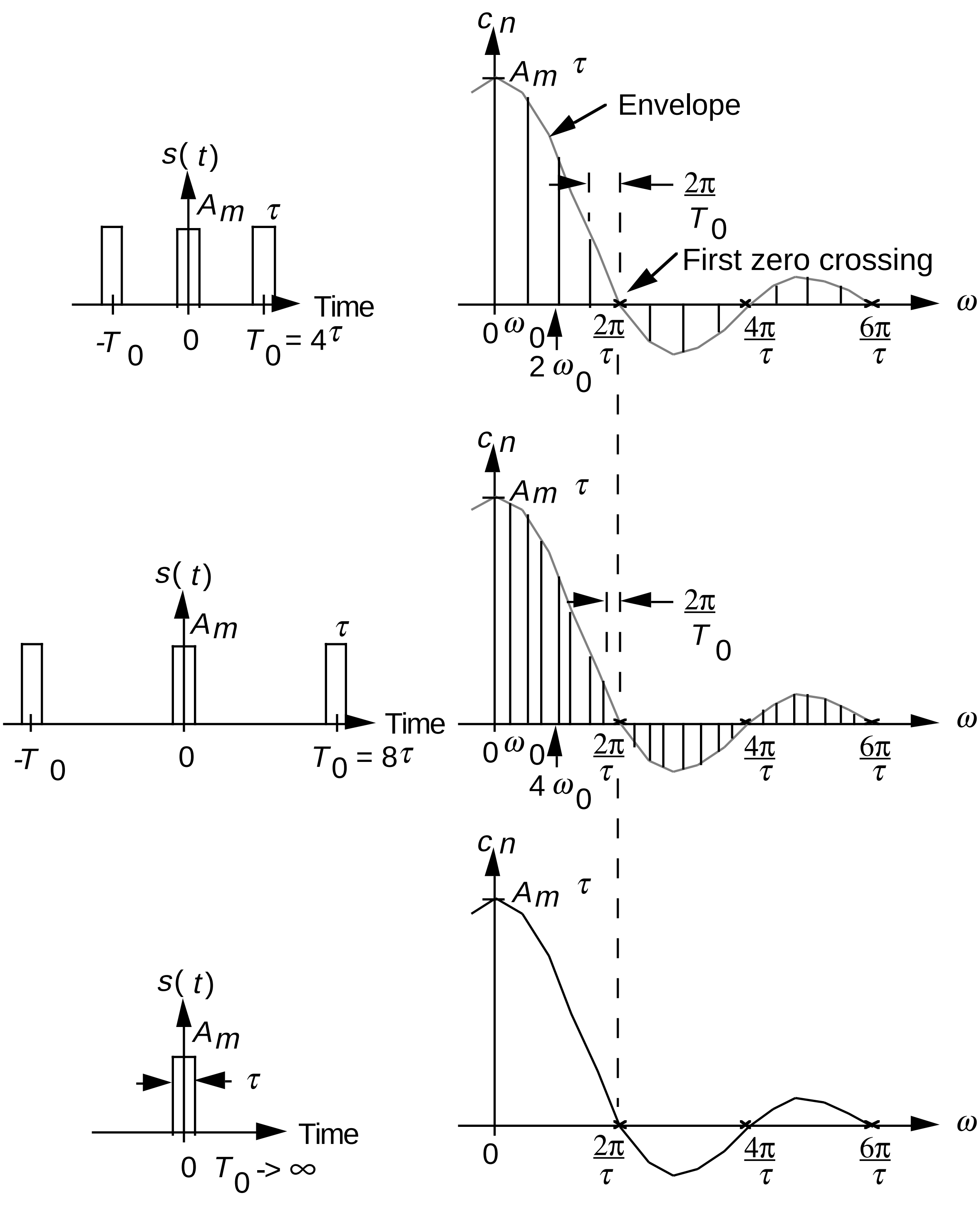

Figure 3.1 Effect on frequency spectrum of increasing period T_0.

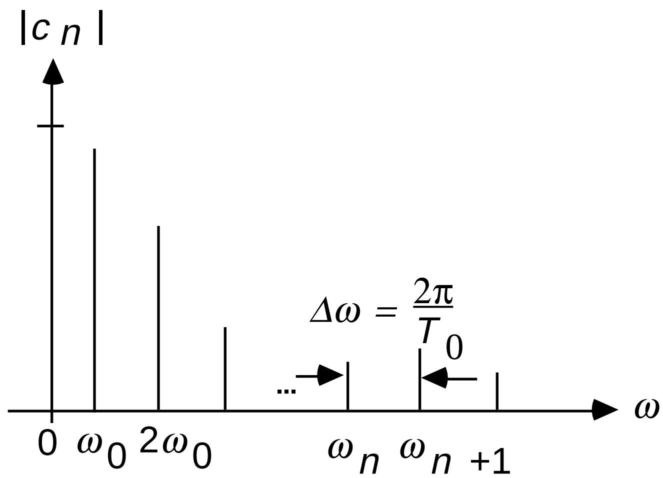

Figure 3.2 Amplitude spectrum of a periodic time function.

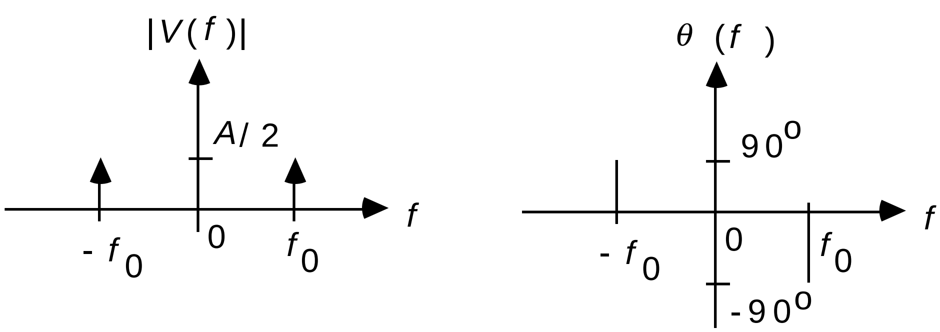

Figure 3.3 Spectrum of the periodic function A sin $2\pi f_0 t$.

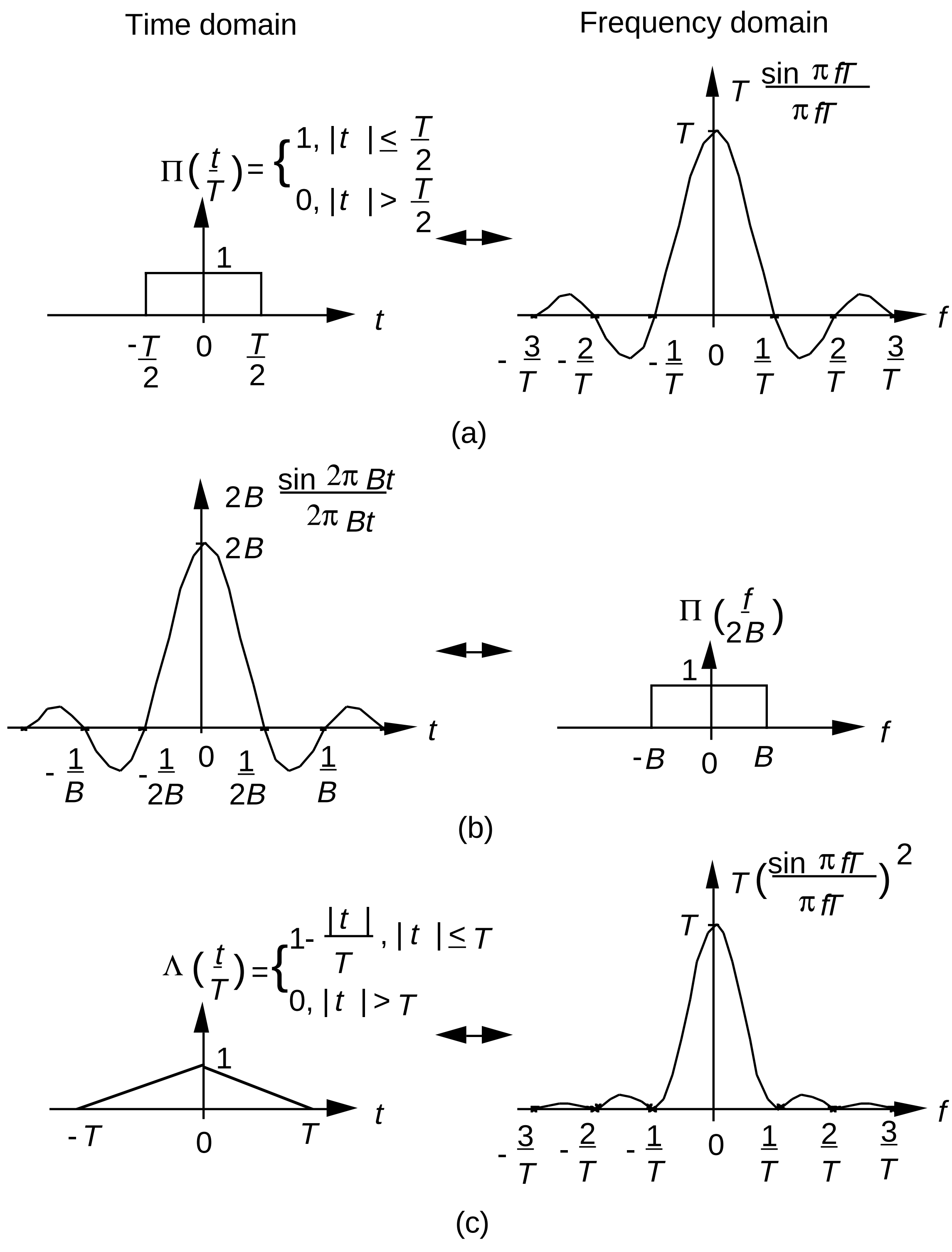

Figure 3.4 Spectra of (a) rectangular, (b) sin *x*/*x*, and (c) triangular pulses.

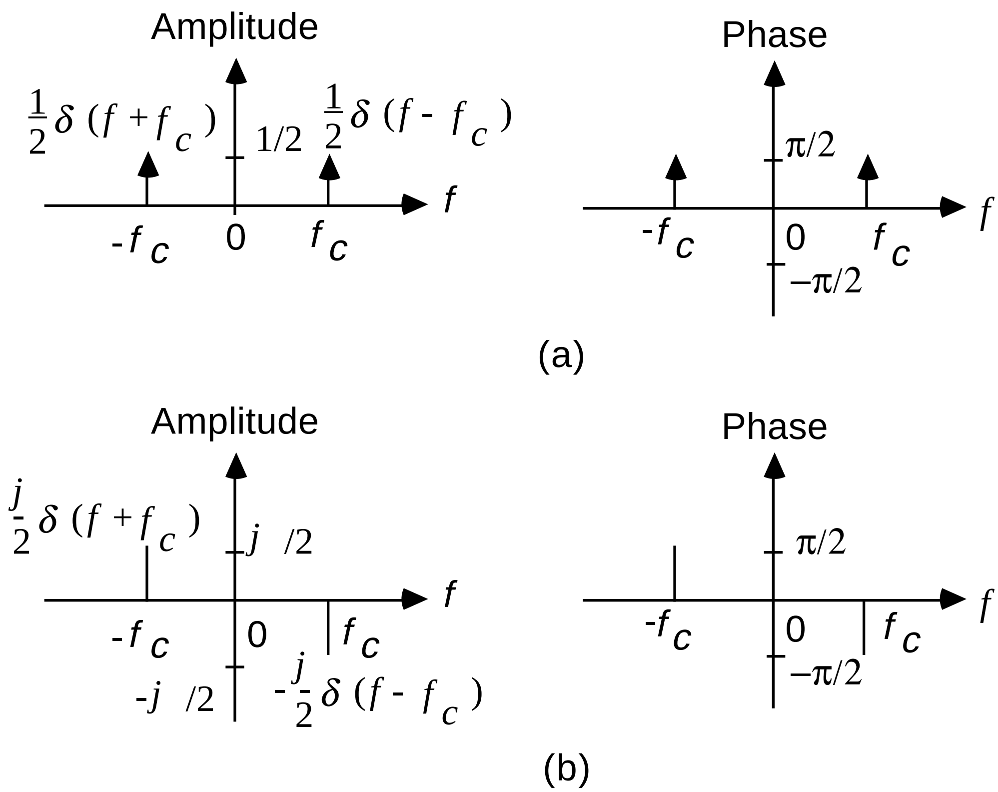

Figure 3.5 Fourier transform spectrum of (a) cos $2\pi f_c t$, and (b) sin $2\pi f_c t$.

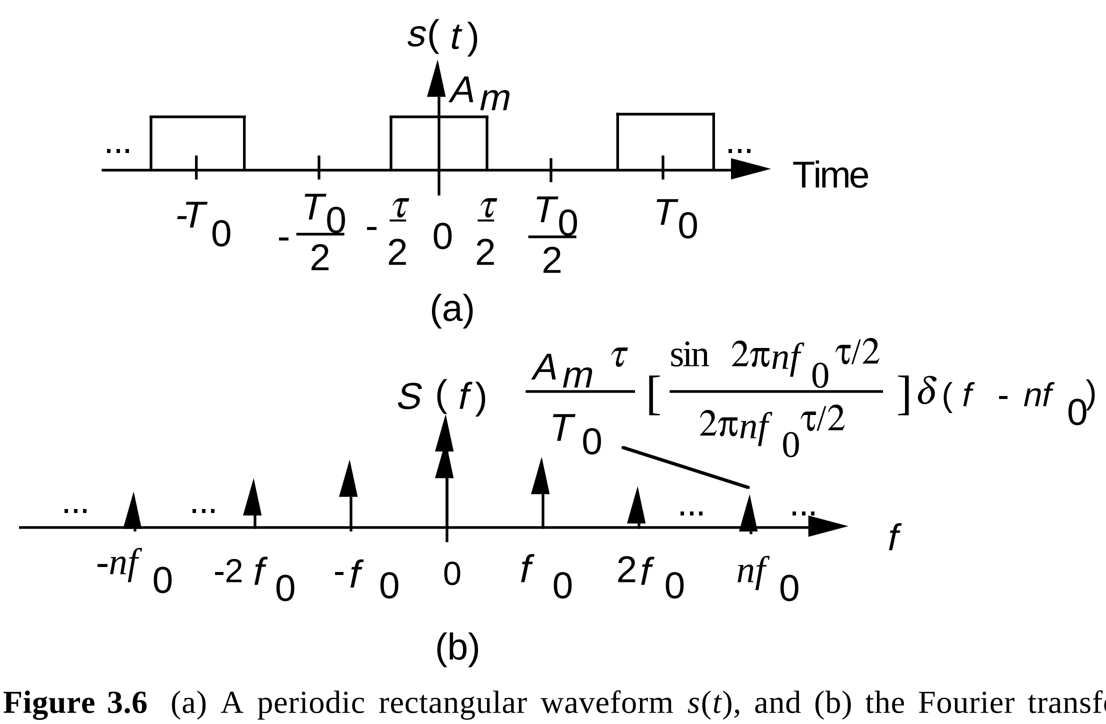

Figure 3.6 (a) A periodic rectangular waveform $s(t)$, and (b) the Fourier transform spectrum of $s(t)$.

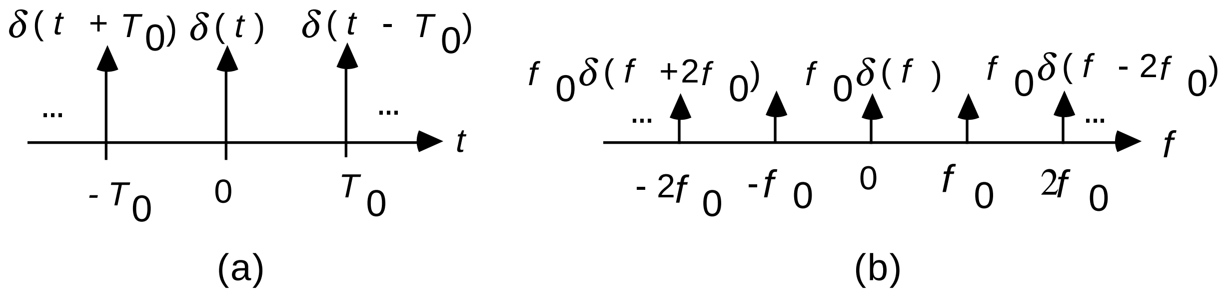

Figure 3.7 (a) Periodic impulse $s(t)$, and (b) Fourier transform spectrum of $s(t)$.