21. Orthonormal Representation of Signals

Introduction

An analogue communication system is designed for the transmission of information in analogue form. The source information is in analogue form. In practice, the communication channel is an analogue channel. At the sending end of the channel, the analogue information (the modulating signal) is used to modulate a *carrier*. The modulated signal is now suitable for transmission over the channel. The operation performed on the analogue signal is called *analogue modulation*. Many analogue communication systems are still in wide use today. These include AM, FM, and PM systems. The growth of digital computers has created the need for digital systems. Broadly speaking, there are two kinds of digital systems. These are :

1. Digital communication system - designed for the transmission of information in digital form. The source information may be an analogue or digital source. An analogue source can be converted into digital form by analogue-to-digital conversion (e.g., pulse-code modulation and delta modulation).

2. Data communication system - designed for the transmission of information in digital form. The source information is already in digital form.

In a data/digital communication system, the digital information is used to modulate a carrier. The operation performed with the digital signal is called digital modulation. At the receiving end, a process of demodulation is used to recover the original signal. Table 21.1 lists a number of digital *modem* (modulation/demodulation) techniques.

Table 21.1 Some digital modem techniques.

We are here concerned with the principles of digital modulation techniques with coherent (with carrier recovery) demodulation/detection of a digital signal in the presence of additive white Gaussian noise (AWGN). In general, the digital signal may be transmitted directly (transmission at baseband) or as a modulated-carrier signal (transmission at radio frequency). In both transmission cases, the concept of signal space can be used to represent a set of signals in terms of a set of orthonormal functions. The Gram-Schmidt orthogonal procedure is therefore discussed. We shall make calculations of *error probability* for various digital modems based on *matched-filter detection* and signal space concepts.

Orthonormal Series Representation of Signals

It is often convenient to represent a set of signals (functions) in terms of a set of *orthonormal basis functions* which is both orthogonal and normalised. All possible linear combinations of the orthonormal basis functions form a linear space known as a *signal space* (function-space coordinate system). The coordinate axes in the signal space are the basis functions $u_1(t)$, $u_2(t)$, ..., $u_n(t)$. Any signal formed from the basis functions can be represented as a point in the signal space and the conventional vector theory applies. The graphical representation of the signals is called a *phasor* or a *signal constellation diagram* and the coefficients that represent signal $s_i(t)$ can be written by a vector

$$S_i = [s_{i1}\ \ s_{i2}\ \dots\ \ s_{in}]. \tag{21.1}$$

Example 21.1

Figure 21.1 2-dimensional signal space.

Gram-Schmidt Orthogonal Process

Any set of *finite energy* signals (functions) can be represented by a set of orthonormal basis functions. These basis functions $u_1(t)$, $u_2(t)$, ..., $u_n(t)$ are derived from the signals $s_1(t)$, $s_2(t)$, ..., $s_m(t)$ where

$$s_i(t) = \sum_{j=1}^{n} s_{ij}\, u_j(t) \tag{21.2}$$

and the mathematical definition of an orthonormal set over the interval (0, T) is

$$\int_0^T u_j(t)\, u_k(t)\, dt = \begin{cases} 1 & if \quad j = k \\ 0 & if \quad j \neq k \end{cases} \tag{21.3}$$

The method by which the basis set is generated is called the Gram-Schmidt process and is as follows. Let the signal set be $\{s_i(t)\}$, i = 1, 2, ..., m, and the *orthonormal basis* set be $\{u_j(t)\}$, j = 1, 2, ..., n for $m \leq n$. In most cases, it is convenient to let $m = n$.

1. Set s_{ij} = 0 except s_{11} in equation (21.2). We then have

$$\int_0^T [s_1(t)]^2\, dt = \int_0^T s_{11}^2\, u_1(t)\, u_1(t)\, dt$$

$$\left\{ \int_0^T [s_1(t)]^2 \, dt \right\}^{1/2} = s_{11} \tag{21.4}$$

and we can find

$$u_1(t) = s_1(t)/s_{11}. \tag{21.5}$$

2. Set $s_{ij} = 0$ except s_{21} and s_{22} in equation (21.2). We then have

$$s_2(t) = s_{21}u_1(t) + s_{22}\, u_2(t) \tag{21.6}$$

$$\int_0^T s_2(t)\, u_1(t)\, dt = \int_0^T s_{21}\, [u_1(t)]^2 \, dt + \int_0^T s_{22}\, u_2(t)\, u_1(t)\, dt$$

$$\int_0^T s_2(t)\, u_1(t)\, dt = s_{21} + 0 \tag{21.7}$$

and we can now evaluate s_{22}. Equation (21.6) can be rewritten as

$$s_2(t) - s_{21}\, u_1(t) = s_{22}\, u_2(t). \tag{21.8}$$

Squaring and integrating, we have

$$\int_0^T [s_2(t) - s_{21}\, u_1(t)]^2 \, dt = s_{22}{}^2 \int_0^T [u_2(t)]^2 \, dt$$

$$\left\{ \int_0^T [s_2(t) - s_{21}\, u_1(t)]^2 \, dt \right\}^{1/2} = s_{22} \tag{21.9}$$

and we can use equation (21.6) to find

$$u_2(t) = [s_2(t) - s_{21}\, u_1(t)]/s_{22} \tag{21.10}$$

3. The process is continued in the same manner until we have found m orthonormal basis functions $u_1(t)$, $u_2(t)$, ..., $u_m(t)$. Given m linearly independent signals $s_1(t)$, $s_2(t)$, ..., $s_m(t)$, the process will generate $m \leq n$) orthonormal basis functions. If the process had been started with a different signal initially (e.g., $s_3(t)$), the basis functions would have been different.

Summary

1. Given $s_i(t)$ for $i = 1, 2, ..., m$, the i-th signal is

$$s_i(t) = s_{i1}u_1(t) + s_{i2}u_2(t) + ... + s_{i(j=i)}\, u_j(t) \qquad (21.11)$$

2. At the $(j = i)$-th step, $u_1(t)$, $u_2(t)$, ..., $u_{j-1}(t)$ are known. We then find s_{i1}, s_{i2}, ..., s_{ij}, where

$$s_{i1} = \int_0^T s_i(t)\, u_1(t)\, dt \qquad (21.12a)$$

$$s_{i2} = \int_0^T s_i(t)\, u_2(t)\, dt \qquad (21.12b)$$

:

$$s_{ij} = \{ \int_0^T [s_i(t) - s_{i1}u_1(t) - ... - s_{ij-1}\, u_{j-1}(t)]^2\, dt\}^{1/2} \qquad (21.12j)$$

3. Find $u_{(j=i)}(t)$ from equation (21.11).

Example 21.2

Given $s_1(t) = 2$ for $0 \le t \le T$, elsewhere $= 0$ and $s_2(t) = 4$ for $0 \le t \le T/2$, elsewhere $= 0$, find s_{11}, s_{21}, and s_{22}.

$$s_{11} = \{ \int_0^T [s_1(t)]^2\, dt\}^{1/2} = \{ \int_0^T 2^2\, dt\}^{1/2} = 2\sqrt{T}\ .$$

$$u_1(t) = s_1(t)/s_{11} = 2/\sqrt{4T}\ .$$

$$s_{21} = \int_0^T s_2(t)u_1(t)\, dt = \int_0^{T/2} 4[s_1(t)/s_{11}]\, dt = 2\sqrt{T}\ .$$

$$s_{22} = \{ \int_0^T [s_2(t) - s_{21}\, u_1(t)]^2\, dt\}^{1/2}$$

$$= \left\{ \int_0^T [4 - 2\sqrt{T}\ (2/\sqrt{4T})]^2\, dt \right\}^{1/2}$$

$$= 2\sqrt{T}\ .$$

$$s_1(t) :\ S_1 = [s_{11}\ \ s_{12}] = [2\sqrt{T}\quad 0]$$
$$s_2(t) :\ S_2 = [s_{21}\ \ s_{22}] = [2\sqrt{T}\quad 2\sqrt{T}\ ].$$

Reference

[1] Taub, H. and Schilling, D. L., Principles of Communication Systems, 2/e, McGraw-Hill, 1987.

Modem	Description
MASK	M-ary amplitude-shift keying
OOK	On-off keying
MPSK	M-ary phase-shift keying
DE-MPSK	Differentially encoded, coherent MPSK
DMPSK	Differential MPSK (no carrier recovery)
OQPSK/ SQPSK	Offset quaternary PSK/Staggered QPSK
TSI-OQPSK	Two-symbol interval OQPSK
M-ary FSK	MFSK
MSK/FFSK	Minimum-shift keying/Fast FSK
DMSK	Differential MSK
GMSK	Gaussian MSK
TFM	Tamed frequency modulation
Multi-h FM	Multi-index; correlative; duobinary FM
CPFSK	Continuous-phase FSK
SFSK	Sinusoidal FSK
M-QAM	M-point quadrature amplitude modulation
SM-QAM	Superimposed M-point QAM

Table 21.1 Some digital modem techniques.

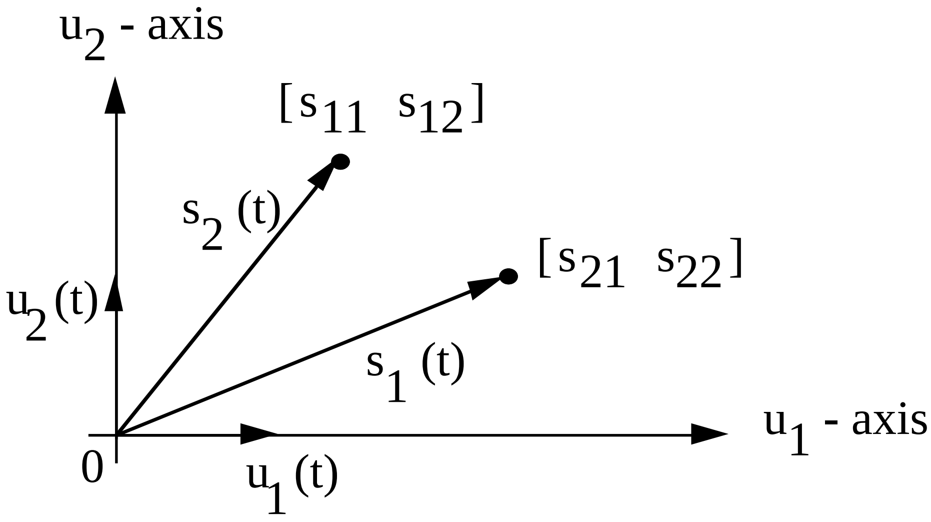

Figure 21.1 2-dimensional signal space.