

4. Classification of Systems, Impulse Response, and Transfer Function

A system is a mathematical model that relates the output signal to the input signal of a physical process.

Figure 4.1 Representation of a system.

Classification of Systems [1]

1. *Linear* and *non-linear* systems

Let $x_i(t)$ and $y_i(t)$, $i \geq 1$, be input and output signals of a system, respectively. A system is called a *linear* system if the input $x_1(t) + x_2(t) + \dots + x_i(t) + \dots$ produces a response $y_1(t) + y_2(t) + \dots + y_i(t) + \dots$, and $ax_i(t)$ produces $ay_i(t)$ for all input signals $x_i(t)$ and scalar a . This is known as the *superposition theorem* and a linear system obeys this principle.

Figure 4.2 Linear system.

In practice, it may be found that a system is only linear over a limited range of input signals.

A *non-linear* system does not obey the superposition theorem.

2. *Causal* and *non-causal* systems

Let $x(t)$ and $y(t)$ be the input and output signals of a system. A *causal* (physically realisable) system produces an output response at time t_1 for an input at time t_0 , where $t_0 \geq 0$ and $t_0 \leq t_1$.

Figure 4.3 Signals associated with causal system.

In other words, a causal system is one whose response does not begin before the input signal is applied.

A *non-causal* system response will begin before the input signal is applied. It can be made realisable by introducing a positive time delay into the system.

3. *Time-invariant* and *time-varying* systems

If the input $x(t - t_0)$ produces a response $y(t - t_0)$ where t_0 is any real constant,

the system is called a *time-invariant* system.

Figure 4.4 Time-invariant system.

If the above condition is not satisfied, the system is called a *time-varying* system.

A system is called a *linear time-invariant (LTI)* system if the system is linear and time-invariant.

Classification of signals and systems will help us in finding a suitable mathematical model for a given physical process that is to be analysed.

Impulse Response

The *impulse response* $h(t)$ of an LTI system is defined as the response of the system when the input signal $x(t)$ is a delta function $\delta(t)$. The output $y(t)$ of an LTI system can be expressed as the convolution of the input signal $x(t)$ and the impulse response $h(t)$ of the system, i.e.,

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\lambda) h(t-\lambda) d\lambda \quad (4.1)$$

$$y(t) = h(t) * x(t) = \int_{-\infty}^{\infty} h(\lambda) x(t-\lambda) d\lambda \quad (4.2)$$

For a causal LTI system,

$$y(t) = \int_0^{\infty} x(\lambda) h(t-\lambda) d\lambda = \int_0^{\infty} h(\lambda) x(t-\lambda) d\lambda \quad (4.3)$$

Transfer Function

In the frequency domain, the Fourier transform of $y(t) = h(t) * x(t)$ is

$$Y(f) = H(f) X(f) \quad (4.4)$$

where $H(f)$ is the Fourier transform of $h(t)$. $H(f)$ is called the *transfer function* or *frequency response* of the LTI system.

Example 4.1 An RC circuit is shown in Figure 4.5. Using Kirchoff's voltage law, we get

$$x(t) = i(t)R + y(t)$$

Since $i(t) = C \frac{dy}{dt}$, we can write

$$RC \frac{dy}{dt} + y(t) = x(t)$$

and

$$RC(j2\pi f)Y(f) + Y(f) = X(f)$$

The transfer function of the RC circuit is $H(f) = \frac{1}{1 + j 2\pi RC f}$ and the impulse

$$\text{response is } h(t) = \begin{cases} \frac{1}{RC} e^{-\frac{t}{RC}}, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

Figure 4.5 Characteristics of an RC circuit.

Consider the input signal $x(t) = \delta(t)$. The Fourier transform of $x(t)$ is

$$X(f) = F[\delta(t)] = \int_{-\infty}^{\infty} \delta(t) e^{-j 2\pi f t} dt = 1 \quad (4.5)$$

and equation (4.4) becomes $Y(f) = H(f)$. $X(f) = 1$ implies equal amplitude at all frequencies. It is equivalent to exciting the system with all frequencies simultaneously. Rather than applying a sinusoidal signal and varying its frequency continuously to obtain $H(f)$, a technique to measure $H(f)$ is as follows:

1. Excite the system with $x(t) = \delta(t)$.
2. Measure $y(t) = h(t)$.
3. Find $H(f) = \int_{-\infty}^{\infty} h(t) e^{-j 2\pi f t} dt$.

In general, $H(f)$ is a complex function of frequency. In polar form, $H(f)$ can be expressed as

$$H(f) = |H(f)| e^{j\theta(f)} \quad (4.6)$$

From our earlier study of the properties of Fourier transform, we have seen that if $h(t)$ is real,

$$H(-f) = H^*(f) \quad (4.7)$$

$$|H(-f)| = |H(f)| \quad (4.8)$$

$$\theta(-f) = -\theta(f) \quad (4.9)$$

where $H^*(f)$ is the complex conjugate of $H(f)$. If $x(t)$ and $y(t)$ are real and $H(f)$ is the transfer function of a LTI system, we can obtain the following results.

Let $x(t)$ and $y(t)$ be the input and output **energy** signals of a LTI system. The energy spectral densities of $x(t)$ and $y(t)$ are $E_{xx}(f) = |X(f)|^2$ and $E_{yy}(f) = |Y(f)|^2$, respectively. Since $Y(f) = H(f) X(f)$, we have

$$E_{yy}(f) = |H(f)|^2 E_{xx}(f) \quad (4.10)$$

$$E_{yy}(f) = H^*(f)H(f) E_{xx}(f) \quad (4.11)$$

Let $x(t)$ and $y(t)$ be the input and output **power** signals of a LTI system. Also, let $x_T(t)$ and $y_T(t)$ be the truncated signals of $x(t)$ and $y(t)$, respectively, where

$$x_T(t) = \begin{cases} x(t), & -T/2 < t < T/2 \\ 0, & \text{elsewhere} \end{cases} = x(t)\Pi\left(\frac{t}{T}\right) \quad (4.12)$$

and

$$y_T(t) = \begin{cases} y(t), & -T/2 < t < T/2 \\ 0, & \text{elsewhere} \end{cases} = y(t)\Pi\left(\frac{t}{T}\right) \quad (4.13)$$

$x_T(t)$ and $y_T(t)$ are **energy** signals as long as T is finite. The power spectral densities

of $x(t)$ and $y(t)$ are $P_{xx}(f) = \lim_{T \rightarrow \infty} \left(\frac{|X_T(f)|^2}{T} \right)$ and $P_{yy}(f) = \lim_{T \rightarrow \infty} \left(\frac{|Y_T(f)|^2}{T} \right)$, respectively. Since $Y(f) = H(f) X(f)$, we have

$$P_{yy}(f) = |H(f)|^2 P_{xx}(f) \quad (4.14)$$

$$P_{yy}(f) = H^*(f)H(f) P_{xx}(f) \quad (4.15)$$

We can also obtain the relationship between the input and output auto-correlation functions of a LTI system. Since the energy/power spectral density and the auto-correlation function are Fourier transform pairs, the inverse Fourier transform of $E_{yy}(f)$ or $P_{yy}(f)$ is [2]

$$R_{yy}(\tau) = h(-\tau) * h(\tau) * R_{xx}(\tau) \quad (4.16)$$

where $R_{xx}(\tau)$ and $R_{yy}(\tau)$ are the auto-correlation functions of $x(t)$ and $y(t)$, respectively.

Figure 4.6 Input and output relationships of linear system.

Distortionless Transmission [1]

In communication systems, a distortionless transmission is often desired. This implies that the output signal $y(t)$ is given by

$$y(t) = K x(t - t_d) \quad (4.17)$$

where K is a constant and t_d is a time delay. The Fourier transform of $y(t)$ is

$$Y(f) = K e^{-j2\pi f t_d} X(f) \quad (4.18)$$

$$= H(f) X(f) \quad (4.19)$$

where $H(f) = K e^{-j2\pi f t_d}$. Figure 4.7 shows the waveforms and spectra associated with distortionless transmission.

Figure 4.7 Waveforms and spectra associated with distortionless transmission.

Classification of Filters [1]

1. Ideal Low-Pass Filter.

The transfer function of an ideal low-pass filter is defined by

$$H_{LPF}(f) = \begin{cases} e^{-j2\pi f t_d} & \text{for } |f| \leq f_c \\ 0 & \text{elsewhere} \end{cases} \quad (4.20)$$

Figure 4.8 Frequency response of an ideal low-pass filter.

The bandwidth of an ideal low-pass filter is equal to f_c .

2. Ideal High-Pass Filter.

The transfer function of an ideal high-pass filter is defined by

$$H_{HPPF}(f) = \begin{cases} e^{-j2\pi f t_d} & \text{for } |f| \geq f_c \\ 0 & \text{elsewhere} \end{cases} \quad (4.21)$$

or

$$H_{HPPF}(f) = e^{-j2\pi f t_d} - H_{LPPF}(f) \quad (4.22)$$

Figure 4.9 Frequency response of an ideal high-pass filter.

The bandwidth of an ideal high-pass filter is not defined, or infinite.

3. Ideal Bandpass Filter.

The transfer function of an ideal bandpass filter is defined by

$$H_{BPPF}(f) = \begin{cases} e^{-j2\pi f t_d} & \text{for } f_{c1} \leq |f| \leq f_{c2} \\ 0 & \text{elsewhere} \end{cases} \quad (4.23)$$

Figure 4.10 Frequency response of an ideal bandpass filter.

The bandwidth of an ideal bandpass filter is equal to $f_{c2} - f_{c1}$.

4. Ideal Bandstop Filter.

The transfer function of an ideal bandstop filter is defined by

$$H_{BSPF}(f) = \begin{cases} e^{-j2\pi f t_d} & \text{for } 0 \leq |f| \leq f_{c1} \text{ and } |f| \geq f_{c2} \\ 0 & \text{elsewhere} \end{cases} \quad (4.24)$$

or

$$H_{BSPF}(f) = e^{-j2\pi f t_d} - H_{BPPF}(f) \quad (4.25)$$

Figure 4.11 Frequency response of an ideal bandstop filter.

The bandwidth of an ideal bandstop filter is not defined, or infinite.

Concept of Non-Ideal or Practical Filter (System) Bandwidth [1]

A common definition of a non-ideal filter's bandwidth is the 3dB bandwidth.

For low-pass filters, the bandwidth is defined as the positive frequency at which the amplitude spectrum $|H(f)|$ drops to a value equal to $|H(f)|/\sqrt{2}$.

For bandpass filters, the bandwidth is defined as the difference between the frequencies at which the amplitude spectrum $|H(f)|$ drops to a value equal to $|H(f_0)|/\sqrt{2}$, where $H(f_0)$ is the peak value of $|H(f)|$.

Figure 4.12 Non-ideal filter bandwidth.

References

- [1] H. P. Hsu, Analog and Digital Communications, McGraw-Hill, 1993.
- [2] L. W. Couch II, Digital and Analog Communication Systems, 5/e, Prentice Hall, 1997.

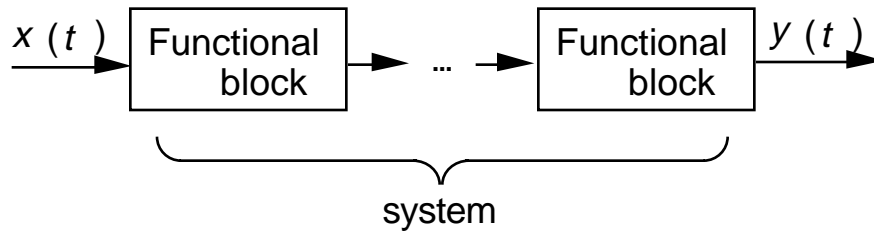


Figure 4.1 Representation of a system.

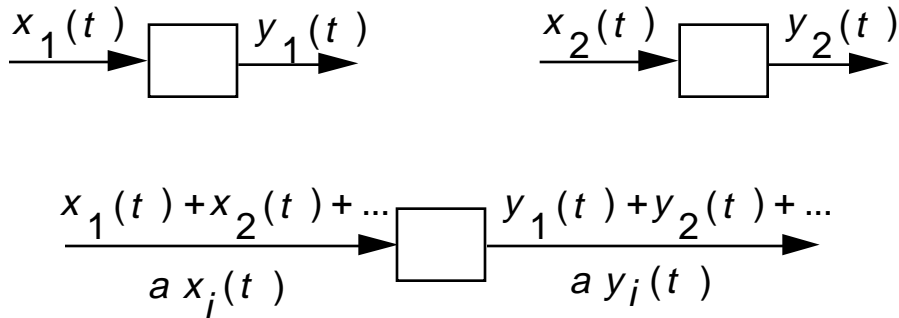


Figure 4.2 Linear system.

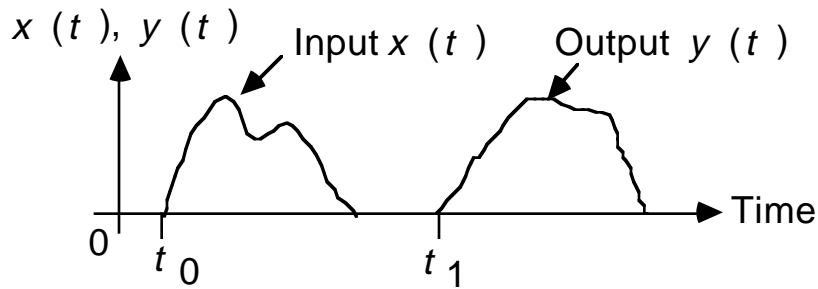


Figure 4.3 Signals associated with causal system.

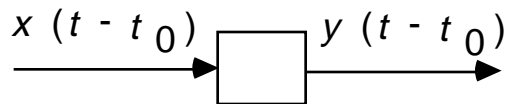


Figure 4.4 Time-invariant system.

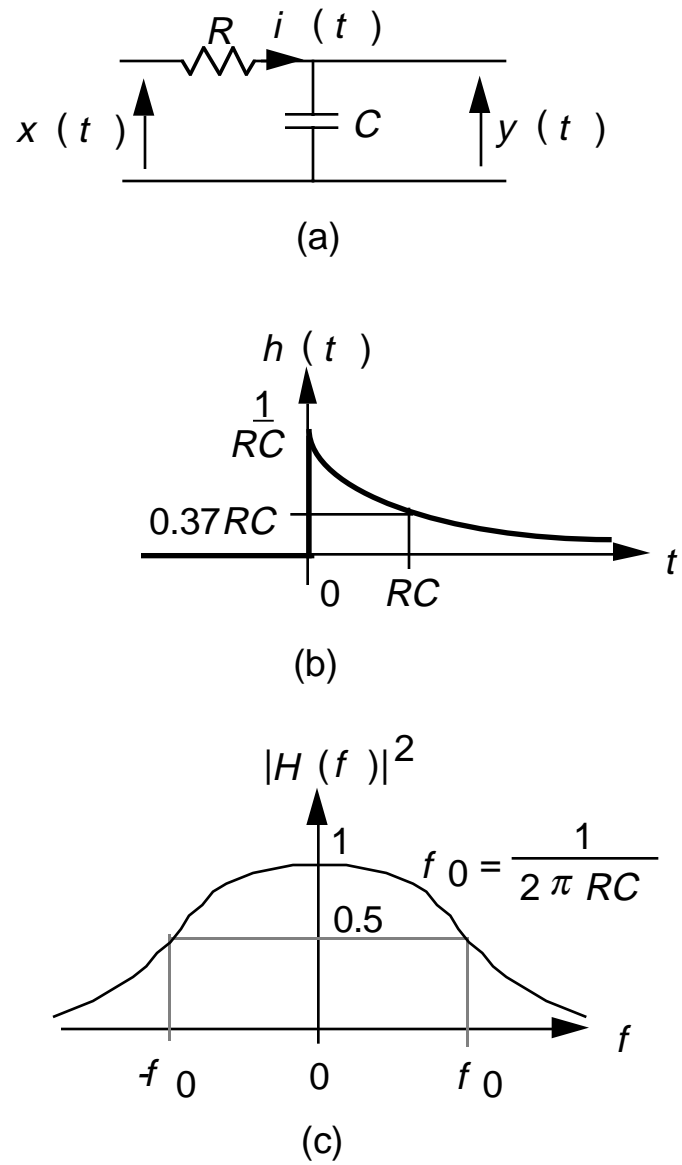


Figure 4.5 Characteristics of an RC circuit.

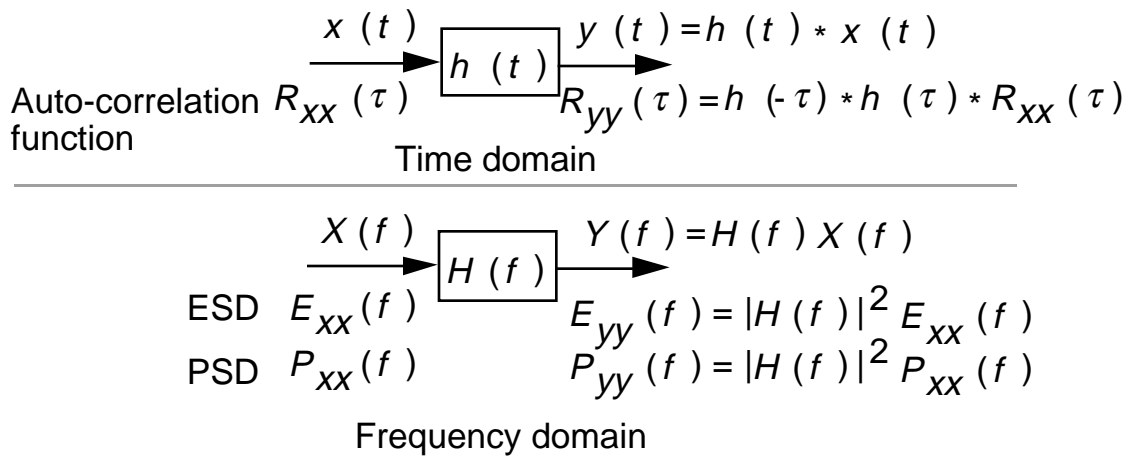


Figure 4.6 Input and output relationships of linear system.

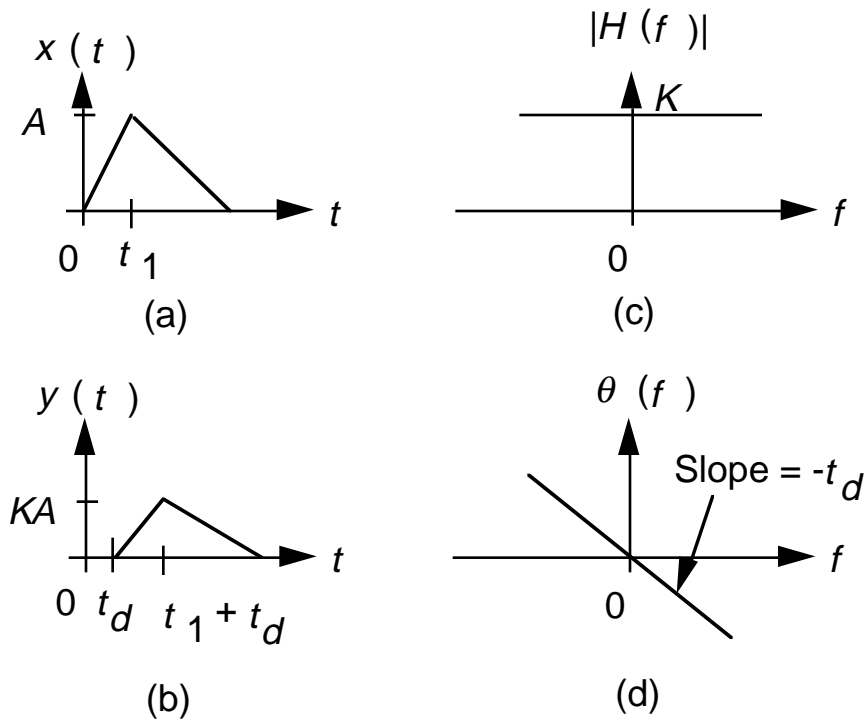


Figure 4.7 Waveforms and spectra associated with distortionless transmission.

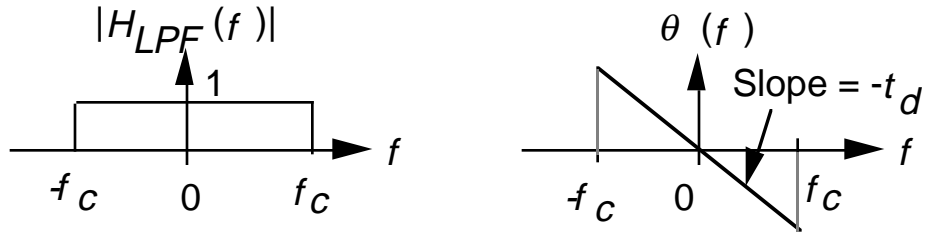


Figure 4.8 Frequency response of an ideal LPF.

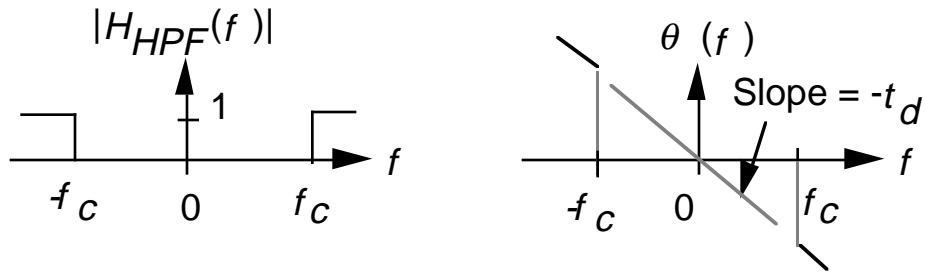


Figure 4.9 Frequency response of an ideal HPF.

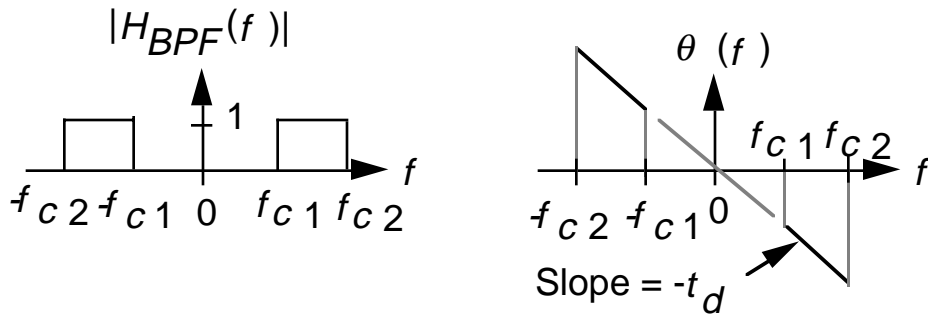


Figure 4.10 Frequency response of an ideal BPF.

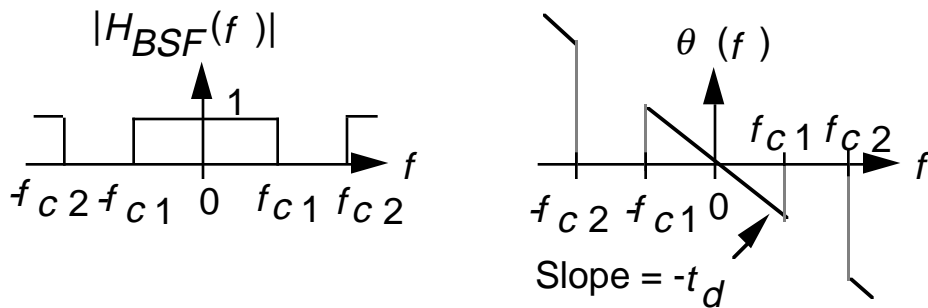


Figure 4.11 Frequency response of an ideal BSF.

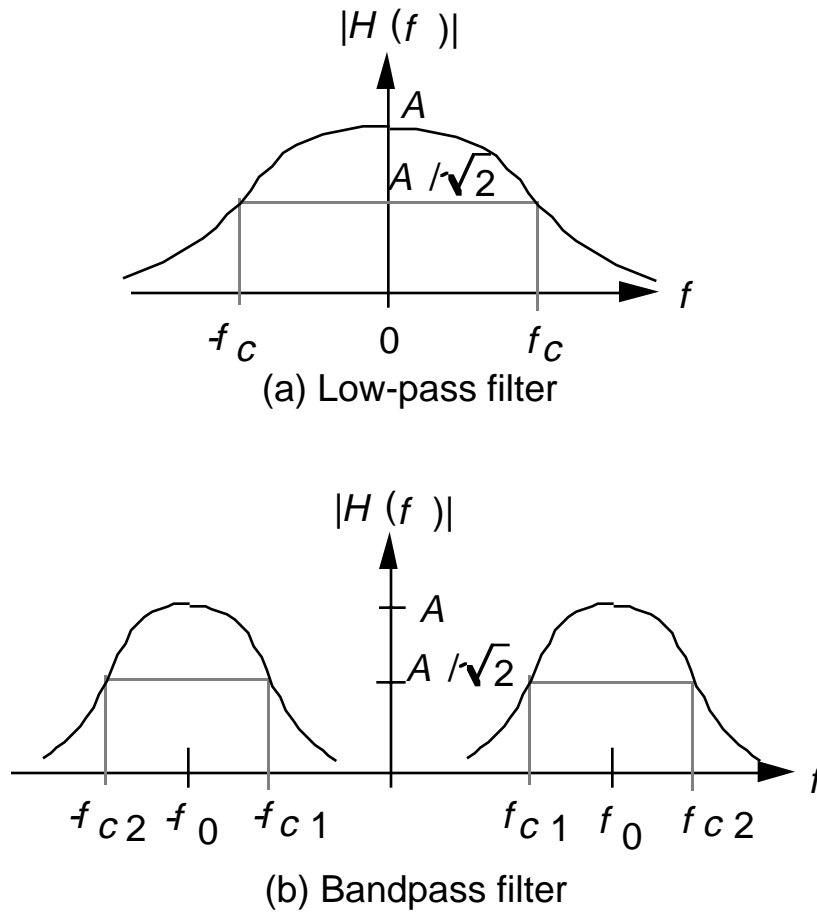


Figure 4.12 Filter bandwidth.