27. Matched-Filter Detection

Random Signals Through Linear Systems

Consider the effect of passing a random signal (message or noise) through a linear system with a transfer function $H(f)$, as shown in Figure 27.1 (a). Let $G_i(f)$ and $R_{ii}(\tau)$ be the power spectral density and the auto-correlation function of the input signal $s_i(t)$, respectively. Also, let $G_o(f)$ and $R_{oo}(\tau)$ be the power spectral density and the auto-correlation function of the output signal $s_o(t)$, respectively. The output power spectral density is related to the input power spectral density by

$$G_o(f) = G_i(f)\, |H(f)|^2 \tag{27.1}$$

Figure 27.1 Handling of signals in a linear system. (a) Random signals, (b) Deterministic signals.

Proof.

Let $h(t)$ be the impulse response of the linear system. The output signal is

$$s_o(t) = \int_{-\infty}^{\infty} h(\tau)\, s_i(t-\tau)\, d\tau = \int_{-\infty}^{\infty} h(t-\tau)\, s_i(\tau)\, d\tau$$

The auto-correlation function of a real-valued power signal $s_o(t)$ is defined by

$$\begin{aligned}
R_{oo}(\tau) &= E[\, s_o(t)\, s_o(t+\tau)] \\
&= E[\int_{-\infty}^{\infty} h(\varepsilon_1)\, s_i(t-\varepsilon_1)\, d\varepsilon_1 \int_{-\infty}^{\infty} h(\varepsilon_2)\, s_i(t+\tau-\varepsilon_2)\, d\varepsilon_2] \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E[s_i(t-\varepsilon_1)\, s_i(t+\tau-\varepsilon_2)]\;\; h(\varepsilon_1)h(\varepsilon_2)\, d\varepsilon_1\, d\varepsilon_2 \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E[s_i(t)\, s_i(t+\tau+\varepsilon_1-\varepsilon_2)]\;\; h(\varepsilon_1)h(\varepsilon_2)\, d\varepsilon_1\, d\varepsilon_2 \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_{ii}(\tau+\varepsilon_1-\varepsilon_2)\;\; h(\varepsilon_1)h(\varepsilon_2)\, d\varepsilon_1\, d\varepsilon_2
\end{aligned}$$

and the Fourier transform of $R_{oo}(\tau)$ is

$$\begin{aligned}
G_o(f) &= \int_{-\infty}^{\infty} R_{oo}(\tau)\;\, e^{-j2\pi f\tau}\, d\tau \\
&= \int_{-\infty}^{\infty} [\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_{ii}(\tau+\varepsilon_1-\varepsilon_2)h(\varepsilon_1)h(\varepsilon_2)d\varepsilon_1 d\varepsilon_2] e^{-j2\pi f\tau}\, d\tau
\end{aligned}$$

$$= \int_{-\infty}^{\infty} h(\varepsilon_1) \int_{-\infty}^{\infty} h(\varepsilon_2) \int_{-\infty}^{\infty} R_{ii}(\tau+\varepsilon_1-\varepsilon_2)e^{-j2\pi f\tau}\, d\tau\, d\varepsilon_1\, d\varepsilon_2$$

Let $\varepsilon = \tau + \varepsilon_1 - \varepsilon_2$, $d\varepsilon = d\tau$

$$\begin{aligned}
G_o(f) &= \int_{-\infty}^{\infty} h(\varepsilon_1) \int_{-\infty}^{\infty} h(\varepsilon_2) \int_{-\infty}^{\infty} R_{ii}(\varepsilon)e^{-j2\pi f(\varepsilon-\varepsilon_1+\varepsilon_2)}\, d\varepsilon\ \ d\varepsilon_1 d\varepsilon_2 \\
&= \int_{-\infty}^{\infty} h(\varepsilon_1)e^{j2\pi f\varepsilon_1}d\varepsilon_1 \int_{-\infty}^{\infty} h(\varepsilon_2)e^{-j2\pi f\varepsilon_2}d\varepsilon_2 \\
&\quad \int_{-\infty}^{\infty} R_{ii}(\varepsilon)e^{-j2\pi f\varepsilon}d\varepsilon \\
&= H^*(f)H(f)G_i(f) \\
&= G_i(f)\,|H(f)|^2
\end{aligned}$$

Q.E.D.

Example 27.1

Suppose the transfer function of a linear filter is

$$\begin{aligned}
H(f) &= 1, \qquad |f| \le B \\
&= 0, \qquad |f| > B
\end{aligned}$$

and the input noise power spectral density is given by

$$G_i(f) = \frac{n_0}{2}$$

The noise power spectral density at the output of the filter is

$$\begin{aligned}
G_o(f) &= G_i(f)\,|H(f)|^2 \\
&= \frac{n_0}{2}, \qquad |f| \le B \\
&= 0, \qquad |f| > B
\end{aligned}$$

and the average output noise power is

$$N = \int_{-\infty}^{\infty} G_o(f)\, df = \frac{n_0}{2} \int_{-B}^{B} |H(f)|^2\, df = \frac{n_0}{2}\, 2B = n_0 B.$$

N is directly proportional to B.

Matched-Filter Detection

A matched filter is a linear filter designed to maximise the output signal-to-noise ratio for a given input signal. Suppose that a signal $s(t)$ plus additive white Gaussian noise $n(t)$ is input to an linear time-invariant filter followed by a sampler, as shown in Figure 27.2.

Figure 27.2 Matched-filter detector.

Let $S(f)$ be the Fourier transform of s(t). Paserval's theorem states that the average signal energy is

$$E = \int_{-\infty}^{\infty} s(t)^2 \, dt = \int_{-\infty}^{\infty} |S(f)|^2 \, df \quad (27.2)$$

The signal at the output of the filter is

$$s_o(t) = \int_{-\infty}^{\infty} S(f)H(f) \, e^{j2\pi ft} \, df \quad (27.3)$$

At time $t = t_0$, we let

$$A = |s_o(t)| = \left| \int_{-\infty}^{\infty} S(f)H(f) \, e^{j2\pi ft_0} \, df \right| \quad (27.4)$$

Since the input noise power spectral density is given by

$$G_n(f) = n_0/2 \quad (27.5)$$

the noise power spectral density at the output of the filter is

$$G_{n_o}(f) = G_n(f) \, |H(f)|^2 = \frac{n_0}{2} \, |H(f)|^2 \quad (27.6)$$

and the average output noise power is

$$N = \int_{-\infty}^{\infty} G_{n_o}(f) \, df = \frac{n_0}{2} \int_{-\infty}^{\infty} |H(f)|^2 \, df \quad (27.7)$$

At $t = t_0$, we have

$$\frac{A^2}{EN} = \frac{\left| \int_{-\infty}^{\infty} S(f)H(f)e^{j2\pi f t_0}\,df \right|^2}{\int_{-\infty}^{\infty} |S(f)|^2\,df \, \frac{n_0}{2} \int_{-\infty}^{\infty} |H(f)|^2\,df} \qquad (27.8)$$

Schwarz's inequality states that, given arbitrary complex functions $X(f)$ and $Y(f)$ of a dummy variable f, then

$$\left| \int_{-\infty}^{\infty} X(f)Y(f)df \right|^2 \leq \int_{-\infty}^{\infty} |X(f)|^2 df \int_{-\infty}^{\infty} |Y(f)|^2 df \qquad (27.9)$$

Equality holds when

$$Y(f) = KX^*(f) \qquad (27.10)$$

where K is a constant and $X^*(f)$ is the complex conjugate of $X(f)$.

We can apply Schwarz's inequality in our matched-filter case by letting $X(f) = S(f)\, e^{j2\pi f t_0}$ and $Y(f) = H(f)$, when equation (27.9) may be written as

$$\left| \int_{-\infty}^{\infty} S(f)\, e^{j2\pi f t_0} H(f)df \right|^2 \leq \int_{-\infty}^{\infty} |S(f)|^2 df \int_{-\infty}^{\infty} |H(f)|^2 df \qquad (27.11)$$

Equality holds when

$$\begin{aligned} H(f) &= K\, [S(f)\, e^{j2\pi f t_0}]^* \\ &= K\, S^*(f)\, e^{-j2\pi f t_0} \end{aligned} \qquad (27.12)$$

and equation (27.8) becomes

$$\frac{A^2}{EN} = \frac{2}{n_0}$$

$$\sqrt{\frac{2E}{n_0}} = \frac{A}{\sqrt{N}} \qquad (27.13)$$

Equality yields maximum output *signal-to-noise ratio (SNR)*.

Taking the inverse Fourier transform of $H(f)$ in equation (27.12), we get

$$h(t) = \int_{-\infty}^{\infty} K\, S^*(f)\, e^{j2\pi f(t - t_0)} df$$

For real $h(t)$, $S(f) = S^*(f)$. Thus

$$h(t) = \int_{-\infty}^{\infty} K\, S(f) e^{j2\pi f(t_0 - t)} df$$

$$h(t) = K\, s(t_0 - t) \qquad (27.14)$$

Figure 27.3 shows the characteristics of a matched filter. The impulse response of the matched filter is a delayed version of the mirror image of $s(t)$. In general, the matched filter is not a *physically realisable (causal)* filter.

Figure 27.3 Matched-filter characteristics.

Equivalence of Matched Filter and Correlator

If $r(t) = s(t) + n(t)$ is the received signal to the input of a *causal* matched filter, then the output of the filter can be found by convolving the received signal $r(t)$ with the impulse response of the filter. Therefore

$$s_o(t) + n_o(t) = r(t) * h(t) = \int_0^{\infty} r(\lambda)\, h(t-\lambda)\, d\lambda \qquad (27.15)$$

Substituting equation (27.14) into equation (27.15), we get

$$s_o(t) + n_o(t) = K \int_0^{\infty} r(\lambda)\, s[t_0 - (t-\lambda)]\, d\lambda \qquad (27.16)$$

At $t = t_0$,

$$s_o(t_0) + n_o(t_0) = K \int_0^{t_0} r(\lambda)\, s(\lambda)\, d\lambda \qquad (27.17)$$

The above operation is known as the *correlation* of $r(t)$ and $s(t)$. For this reason, the matched filter is often referred to as a *correlator*. Figure 27.4 shows the block diagram of a correlator. It is an alternative way of synthesising a matched filter.

Figure 27.4 Correlator.

References

[1] H. Taub and D. L. Schilling, Principles of Communication Systems, 2/e, McGraw-Hill, 1986.

[2] H. P. Hsu, Analog and Digital Communications, McGraw-Hill, 1993.

[3] M. S. Roden, Analog and Digital Communication Systems, 3/e, Prentice Hall, 1991.

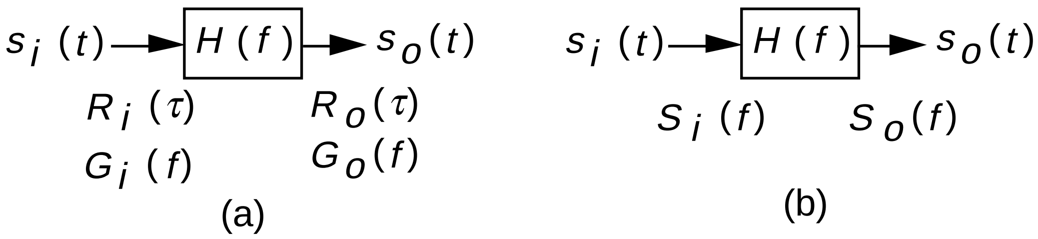

Figure 27.1 Handling of signals in a linear system. (a) Random signals, (b) Deterministic signals.

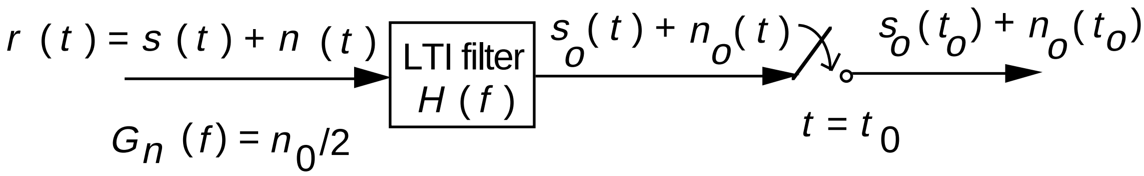

Figure 27.2 Matched-filter detector.

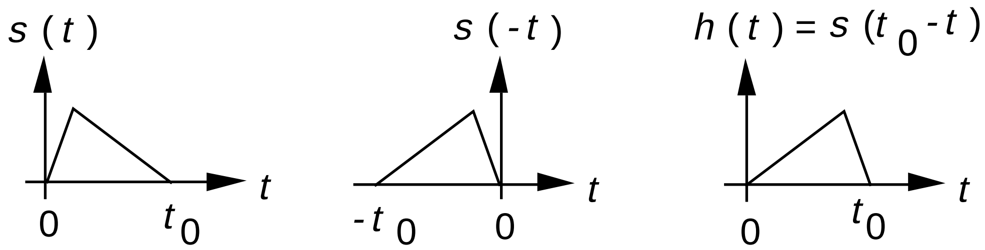

Figure 27.3 Matched-filter characteristics.

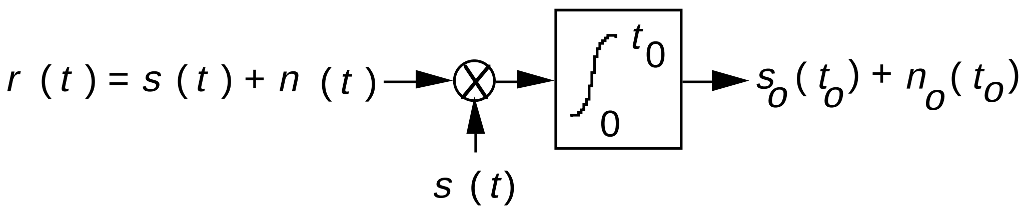

Figure 27.4 Correlator.