# 25. Spectral Representation of Random Signals

In practice, we often deal with signal waveforms of a random nature. Modelling these signals is based on the spectral distribution of the power in the random waveform. A probabilistic model (similar to the concept of random variables) can be used to describe the waveforms in time. This enables us to calculate the error probability of a communication system due to noise which is random in nature. It also enables us to examine the effect of linear filtering.

Consider Figures 25.1 and 25.2.

**Figure 25.1** Random process (bandlimited).

**Figure 25.2** Ensemble averages. (a) Taking from *m* identical sources. (b) Taking from one source.

Here, we simply treat noise as a random process $n(t)$. A random process can be considered as a collection of time functions, $n_1(t)$, $n_2(t)$, ..., which vary in a random manner with time. This time function is called *sample function*. The totality of all sample function is called an *ensemble*. For a specific time $t_1$, $n(t_1)$ is a random variable $n$ with *probability density function* $f_N(n)$. The statistical properties of the random variable $n$ describe the statistical properties of the random process at time $t = t_1$.

To find the probability density function (pdf) of the random variable $n$ (amplitude distribution), we sample $n(t)$ at intervals 'far enough apart' to ensure statistical independence of the samples, and count the number of times the samples fall in the range $n$ and $n + \Delta n$ (lie in that random variable range).

To develop a spectral representation for the noise, we need to measure its mean and variance $\sigma^2$.

## Measurement of Noise Parameters

1. Average Value

From our earlier studies of signals and operations, the time average of a quantity is defined as

$$\langle . \rangle = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} \langle . \rangle dt. \qquad (25.1)$$

We can employ a dc meter to get a reasonable measure of the average value of the quantity. When we use a dc meter to measure the average value of the noise $n(t)$ over a period of $T$ seconds, we can write

$$\bar{n} = \frac{1}{T}\int_0^T n(t)\, dt \tag{25.2}$$

The expected value (mean) of the time average $\bar{n}$ is

$$E[\bar{n}] = E[\frac{1}{T}\int_0^T n(t)\, dt] \tag{25.3}$$

$$= \frac{1}{T}\int_0^T E[n(t)]\, dt \tag{25.4}$$

It is reasonable for us to expect that the dc meter will give a steady reading. In this case, $\bar{n} = E[\bar{n}]$ = constant, $E[n(t)]$ must be independent of time and

$$\bar{n} = E[\bar{n}] = E[n] \tag{25.5}$$

where

$$E[n] = \frac{1}{T}\int_0^T E[n(t)]\, dt \tag{25.6}$$

is the ensemble average of $n(t)$. The time average $\bar{n}$ is equal to the statistical average $E[n]$. This implies that the noise is ergodic, and we can use any time-domain based measuring equipment to obtain the statistical properties of random signals.

## 2. AC Power

To get a reasonable measure of the average ac power (variance) of the noise, we employ a true-rms meter. When we use a rms meter to measure the average power of the noise $n(t)$ over a a period of $T$ seconds, the average power of the noise is

$$P_{av} = \frac{1}{T}\int_0^T n(t)^2\, dt \tag{25.7}$$

The expected value of $P_{av}$ is

$$E[P_{av}] = \frac{1}{T}\int_0^T E[n(t)^2]\,dt \qquad (25.8)$$

Again, it is reasonable for us to expect that the rms meter will give a steady reading. In this case, $P_{av} = E[P_{av}]$ = constant, $E[n(t)^2]$ must be independent of time and

$$P_{av} = E[P_{av}] = E[n^2] \qquad (25.9)$$

where

$$E[n^2] = \frac{1}{T}\int_0^T E[n(t)^2]\,dt \qquad (25.10)$$

is the mean-square value of $n(t)$. The time average power $P_{av}$ is equal to the statistical average power $E[n^2]$. Since the variance

$$\begin{aligned} \sigma^2 &= E\{(n - E[n])^2\} && (25.11)\\ &= E\{(n^2 - 2nE[n] + E^2[n]\}\\ &= E[n^2] - 2E^2[n] + E^2[n]\\ &= E[n^2] - E^2[n] && (25.12) \end{aligned}$$

then $\sigma^2$ must be the ac power of the noise, i.e.,

ac power = average power - square of the dc mean

The variance of the noise can be obtained by subtracting the square of the mean (measured by the dc meter) from the average power (measured by the rms meter).

**Spectral Representation of Random Signals**

How is the noise bandwidth related to the variations in time of $n(t)$? Is it possible to calculate the noise power or the variance $\sigma^2$? The simplest way to explore noise properties is to use an *auto-correlation function* in the time domain and *power spectral density* in the frequency domain. The auto-correlation function of $n(t)$ is defined as

$$R_n(t_1, t_2) = E[n(t_1)\, n(t_2)] \qquad (25.13)$$

$$R_n(\tau) = E[n(t)\, n(t + \tau)] \qquad (25.14)$$

where $\tau = t_2 - t_1$ and

$$R_n(0) = E[n(t)^2] \tag{25.15}$$

When the noise is ergodic, $E[n(t)^2] = E[n^2] = R_n(0)$. Figure 25.3 shows a graphical interpretation of the auto-correlation function.

**Figure 25.3** Auto-correlation definition.

The power spectral density of $n(t)$ is defined as

$$G_n(f) = \int_{-\infty}^{\infty} R_n(\tau)\, e^{-j2\pi f\tau}\, d\tau \tag{25.16}$$

The auto-correlation function and the power spectral density form a Fourier transform pair, i.e.,

$$R_n(\tau) = \int_{-\infty}^{\infty} G_n(f)\, e^{j2\pi f\tau}\, df \tag{25.17}$$

When the noise $n(t)$ is ergodic, $E[n(t)^2] = E[n^2]$ and equation (25.15) can be written as $R_n(0) = E[n^2]$. The total power of the noise is given by

$$R_n(0) = \int_{-\infty}^{\infty} G_n(f)\, df \tag{25.18}$$

Figure 25.4 shows a few examples of spectral density and auto-correlation function pairs.

**Figure 25.4** Examples of spectral density and auto-correlation function pairs.

Observations:

1. Bandwidth is defined in terms of the width of $G_n(f)$.

2. The smaller the bandwidth, the smaller the noise power, and the less fluctuation in time (bandwidth is inversely proportional to $T$).

3. The signal samples becomes less correlated as $\tau$ increases. In Figure 25.4 (b), the signal samples are uncorrelated at multiples of $1/2B$ second intervals.

4. The value of $\tau$ at which $R_n(\tau)$ drops sharply is a measure of the time between significant changes in $n(t)$.

## Analysis of White Noise

White noise has a uniform power spectral density distributed over a very wide range of frequencies up to about $10^{13}$ Hz. The two-sided power spectral density $G_n(f)$ is equal to $n_0/2$. Hence, we obtain for the auto-correlation function from $G_n(f)$:

$$\begin{aligned} R_n(\tau) &= \int_{-\infty}^{\infty} G_n(f)\, e^{j2\pi f\tau}\, df \\ &= n_0/2 \int_{-\infty}^{\infty} e^{j2\pi f\tau}\, df \\ &= (n_0/2)\ \delta(\tau). \end{aligned} \tag{25.19}$$

Since $R_n(\tau)$ has a value at $\tau = 0$ only, there is no correlation between any two samples of white noise separated by an interval $\tau > 0$. Figure 25.5 shows the power spectrum and autocorrelation function of the white noise. For bandlimited white noise, the signal samples are uncorrelated at multiples of $1/2B$ second intervals.

**Figure 25.5** Power spectrum and autocorrelation function. (a) White noise. (b) Bandlimited white noise.

### *Example 25.1*

For white noise, $E[n] = 0$. The average noise power is

$$R_n(0) = \int_{-\infty}^{\infty} G_n(f)\, df = E[n^2]$$

Since $\sigma^2 = E[n^2] - E^2[n]$ and $E[n] = 0$, the average noise power is equal to the variance of the noise.

## Reference

[1] M. Schwartz, Information Transmission, Modulation, and Noise, 4/e, McGraw-Hill, 1990.


**Figure 25.1** Random process (bandlimited).

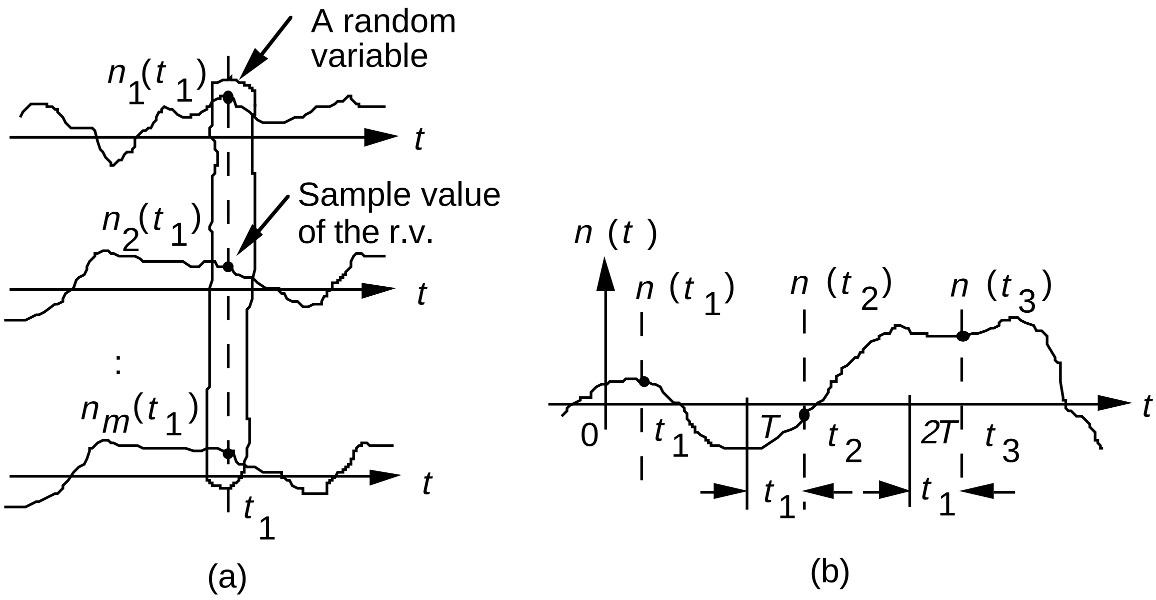



**Figure 25.2** Ensemble averages. (a) Taking from *m* identical sources, (b) Taking from one source.

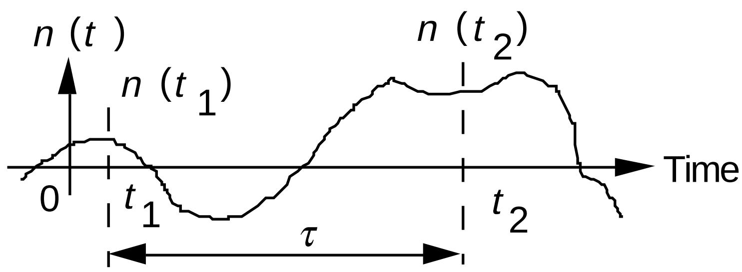



**Figure 25.3** Autocorrelation definition.

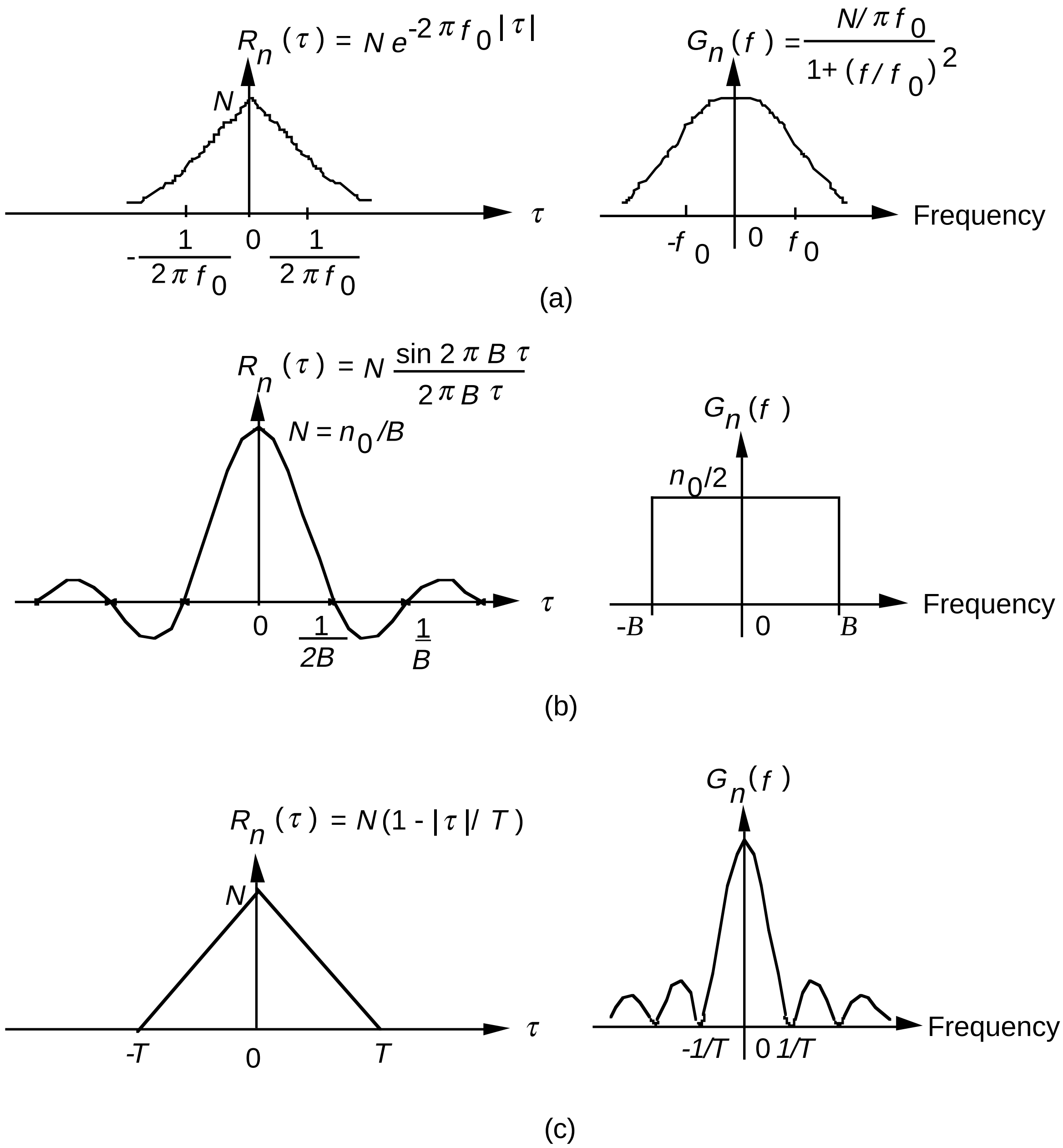
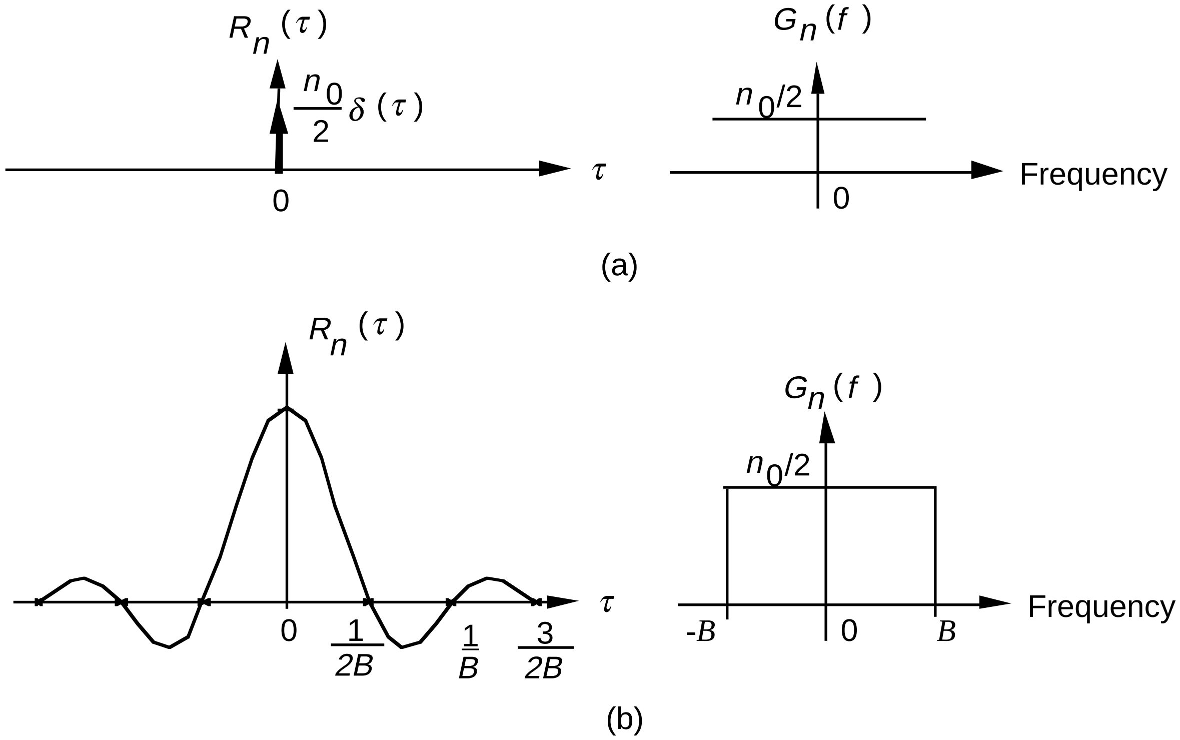



**Figure 25.4** Examples of spectral density and autocorrelation function pairs.

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