12. Nyquist Sampling, Pulse-Amplitude Modulation, and Time-Division Multiplexing

Many analogue communication systems are still in wide use today. These include AM, FM, and PM systems. If analogue signals are to be transmitted digitally, they have to be converted to discrete samples. The conversion of an analogue signal into a discrete-time sampled signal is accomplished by sampling the analogue signal at regular time intervals T_S. T_S is called the *sampling period* and $f_S = 1/T_S$ is known as the *sampling rate*.

Definition [1]. A signal $m(t)$ is called a *band-limited signal* if

$$M(f) = 0 \qquad \text{for } |f| \geq f_m \text{ Hz} \tag{12.1}$$

where f_m is the highest-frequency spectral component of $m(t)$.

Consider a periodic rectangular waveform $s(t)$ of period $T_S = 1/f_S$, unit amplitude, and pulse width τ. The trigonometric Fourier series of $s(t)$ is

$$s(t) = \frac{\tau}{T_S} + \frac{2}{T_S}\sum_{n=1}^{\infty}\left(\tau\frac{\sin\ 2\pi nf_S\tau/2}{2\pi nf_S\tau/2}\right)\cos 2\pi nf_St \tag{12.2}$$

$$= \frac{c_0}{T_S} + \frac{2}{T_S}\sum_{n=1}^{\infty} c_n \cos 2\pi nf_St \tag{12.3}$$

$$= d + 2d\sum_{n=1}^{\infty}\frac{\sin\ \pi nd}{\pi nd}\cos 2\pi nf_St \tag{12.4}$$

where $c_0 = \tau$, $c_n = \tau\frac{\sin\ \pi nd}{\pi nd}$, and $d = \frac{\tau}{T_S}$. The corresponding exponential Fourier series is

$$s(t) = d\sum_{n=-\infty}^{\infty}\frac{\sin\ \pi nd}{\pi nd}e^{j2\pi nf_St} \tag{12.5}$$

If we multiply $m(t)$ by $s(t)$, we obtain

$$\begin{aligned} s_C(t) &= m(t)\,s(t) \\ &= m(t)\left(d + 2d\sum_{n=1}^{\infty}\frac{\sin\ \pi nd}{\pi nd}\cos 2\pi nf_St\right) \\ &= dm(t)\left(1 + 2\sum_{n=1}^{\infty}\frac{\sin\ \pi nd}{\pi nd}\cos 2\pi nf_St\right) \end{aligned} \tag{12.6}$$

$s_C(t)$ consists of the component $m(t)$ and an infinite number of DSB signals at sampling frequencies f_s, $2f_s$, $3f_s$, ... The Fourier transform of $s_c(t)$ is

$$S_c(f) = dM(f) + d \sum_{n=1}^{\infty} \frac{\sin \pi nd}{\pi nd} [M(f - nf_s) + M(f + nf_s)] \tag{12.7}$$

Figure 12.1 shows the waveform and spectra associated with signal sampling.

Figure 12.1 Waveform and spectra associated with signal sampling.

Figure 12.2 shows what happen if $f_s = 2\, f_m$ and $f_s < 2\, f_m$.

Figure 12.2 Signal spectra for (a) $f_s = 2f_m$ and (b) $f_s < 2f_m$.

Since the bandwidth of $m(t)$ is f_m, we see that the spectra do no overlap if $f_s \geq 2\, f_m$ and the spectrum associated with the signal $m(t)$ can be separated from others using a low-pass filter with a cutoff frequency of f_m. When $f_s < 2\, f_m$, the spectra overlap. Since the frequency content in these regions of overlap adds, the signal is distorted. The distortion is called *aliasing* and it is no longer possible to recover $m(t)$ from its sample values by low-pass filtering.

Sampling Theorem [2]

Let $m(nT_s)$ be the sample values of $m(t)$ where n is an integer. The *sampling theorem* states that the signal $m(t)$ can be reconstructed from $m(nT_s)$ with no distortion if *the sampling frequency* $f_s \geq 2f_m$. The minimum sampling rate $2f_m$ is called the *Nyquist sampling rate*.

Proof.

Since $M(f)$ is a non-periodic bandlimited function, we can make a new function which is periodic at frequency f_s but not overlapping in the frequency-domain. Let the periodic frequency function be $M_p(f)$, as shown in Figure 12.3. $M(f)$ is a band-limited version of $M_p(f)$,

Figure 12.3 $M(f)$ represented as a periodic frequency function.

The exponential Fourier series of $M_p(f)$ is

$$M_p(f) = \frac{1}{f_s} \sum_{n=-\infty}^{\infty} c_n\, e^{jnt_0 f} \tag{12.8}$$

where $f_s \geq 2f_m$ and

$$t_0 = 2\pi/f_s \qquad (12.9)$$

The coefficients c_n are given by

$$c_n = \int_{-f_s/2}^{f_s/2} M_p(f) e^{-jnt_0 f} df = \int_{-f_s/2}^{f_s/2} M_p(f)\, e^{-jn(2\pi/f_s)f} df \qquad (12.10)$$

However, the Fourier transform tells us that

$$m(t) = \mathcal{F}^{-1}[M(f)] = \int_{-\infty}^{\infty} M(f)\, e^{j2\pi ft} df = \int_{-f_s/2}^{f_s/2} M_p(f)\, e^{j2\pi ft} df \qquad (12.11)$$

Comparing equations (12.10) and (12.11), we see that, if $t = -\frac{n}{f_s}$, we obtain

$$c_n = m\left(-\frac{n}{f_s}\right) \qquad (12.12)$$

This says that we can obtain each c_n from the sample value of $m(t)$ at time $t = -\frac{n}{f_s}$. Once c_n is known, we can obtain $M_p(f)$ from equation (12.8), and once $M_p(f)$ is known, we can obtain $m(t)$ from equation (12.11). □

Substituting c_n into equation (12.8), we get

$$M_p(f) = \frac{1}{f_s} \sum_{n=-\infty}^{\infty} m\left(-\frac{n}{f_s}\right) e^{jnt_0 f} \qquad (12.13)$$

Substituting this expression for $M_p(f)$ into equation (12.11), we get

$$\begin{aligned} m(t) &= \mathcal{F}^{-1}[M(f)] = \int_{-f_s/2}^{f_s/2} \left[\frac{1}{f_s} \sum_{n=-\infty}^{\infty} m\left(-\frac{n}{f_s}\right) e^{jnt_0 f}\right] e^{j2\pi ft} df \\ &= \sum_{n=-\infty}^{\infty} \frac{1}{f_s} \int_{-f_s/2}^{f_s/2} m\left(-\frac{n}{f_s}\right) e^{jnt_0 f}\, e^{j2\pi ft}\, df \\ &= \sum_{n=-\infty}^{\infty} m\left(-\frac{n}{f_s}\right) \frac{\sin[\pi f_s (t-\frac{n}{f_s})]}{\pi f_s (t-\frac{n}{f_s})} \end{aligned}$$

$$= \sum_{n=-\infty}^{\infty} m(\frac{n}{f_s}) \underbrace{\frac{\sin[\pi f_s(t-\frac{n}{f_s})]}{\pi f_s(t-\frac{n}{f_s})}}_{Weighting\ \ factor} \qquad (12.14)$$

Equation (12.14) shows that each sample is multiplied by a weighting factor.

Signal Reconstruction [2, 3]

The process of reconstructing an analogue signal $m(t)$ from its samples is known as *interpolation*. How do we reconstruct $m(t)$ from its samples $m(-\frac{n}{f_s})$? Consider the sample signal of $m(t)$ shown in Figure 12.4.

Figure 12.4 Samples of $m(t)$.

Let $M_n(f)$ be the Fourier transform of the n-th sample $m(\frac{n}{f_s})$. If $\tau << 1/f_s$, $m(t)$ can be assumed to be constant over the sampling time and

$$M_n(f) = \int_{-\tau/2}^{\tau/2} m(\frac{n}{f_s})e^{-j2\pi ft}dt \approx m(\frac{n}{f_s})e^{-j2\pi f(n/f_s)} \int_{-\tau/2}^{\tau/2} dt$$

$$M_n(f) \approx \tau m(\frac{n}{f_s})e^{-j2\pi f(n/f_s)} \qquad (12.15)$$

From our knowledge of basic PAM theory, we can recover the analogue signal using a low-pass filter with a cutoff frequency of $f_s/2$ ($\geq f_m$). Assume that we have an ideal low-pass filter whose transfer function is $H(f) = K e^{-j2\pi f t_d}$, where K is a constant and t_d is a time delay. Without loss of generality, we set the filter gain $K = 1$ and the filter delay $t_d = 0$.

Let $g_n(t)$ be the filter output response to the n-th input sample $m(\frac{n}{f_s})$. The Fourier transform of $g_n(t)$ is

$$G_n(f) = H(f)M_n(f) = M_n(f)$$

and

$$\begin{aligned} g_n(t) &= \int_{-\infty}^{\infty} G_n(f)\, e^{j2\pi ft}\, df = \int_{-f_s/2}^{f_s/2} G_n(f)\, e^{j2\pi ft}\, df \\ &= \tau\, m\left(\frac{n}{f_s}\right) f_s \frac{\sin[\pi f_s (t - \frac{n}{f_s})]}{\pi f_s (t - \frac{n}{f_s})} \qquad (12.16) \end{aligned}$$

For a linear ideal filter, the filter output response to all input samples is just the sum of the filter output to each input sample, or

$$\begin{aligned} g(t) &= \sum_{n=-\infty}^{\infty} g_n(t) \\ &= \tau f_s \sum_{n=-\infty}^{\infty} m\left(\frac{n}{f_s}\right) \frac{\sin[\pi f_s (t - \frac{n}{f_s})]}{\pi f_s (t - \frac{n}{f_s})} \qquad (12.17) \\ &= \tau\, f_s m(t) \qquad (12.18) \end{aligned}$$

Equation (12.17) yields values of $g(t)$ between samples as a weighted sum of all sample values. $g(t)$ is not only defined at the sampling instants, but it is proportional to $m(t)$ at *all* instants of time. This is shown in Figure 12.5.

Figure 12.5 Filter response to input samples.

Practical Sampling Frequency and Pulse-Amplitude Modulation

Table 12.1 Practical sampling frequency values for audio and broadcast signals

Signal	f_m	$f_s \geq 2f_m$	Actual sampling frequency f_s
Audio	3.3 kHz	≥ 6.6 kHz	8 kHz
Music	20 kHz	≥ 40 kHz	44.1 kHz
TV	4 MHz	≥ 8 MHz	

The sampling theorem is very important because it allows us to replace an analogue signal by a discrete sample and reconstruct the analogue signal from its sample values. It opens doors to many new techniques of communicating analogue signal by samples. A system transmitting sample values of the analogue signal is called a *pulse-amplitude modulation (PAM)* system and is shown in Figure 12.6.

Figure 12.6 PAM system.

Time Division Multiplexing (TDM)

One of the basic problems in communication engineering is the design of a pulse communication system which allows signals from many users to be transmitted simultaneously over a single communication channel. We see from the sampling process that, with $\tau << T_S$, there is a time gap between two consecutive samples in a single-user PAM system. Suppose that we have several different signals of the same or different bandwidth. If we sample the signals in a sequential manner, we can put the samples in the time gaps. All these signal samples can now be transmitted along a single communication channel. At the receiving end, the signals can be separated and recovered. We now have a time-multiplexed system. Such a multiplexing technique is called *time division multiplexing (TDM)*. It permits the simultaneous transmission of several signals on a time-shared basis. Figure 12.7 shows the transmitter, the receiver, and the spectrum of a 5-user TDM PAM system with f_S = 8000 samples/s.

Figure 12.7 TDM system.

References

[1] H. P. Hsu, analog and Digital Communications, McGraw-Hill, 1993.

[2] M. Schwartz, Information Transmission, Modulation, and Noise, 4/e, McGraw-Hill, 1990.

[3] M. S. Roden, Analog and Digital Communication Systems, 3/e, Prentice Hall, 1991.

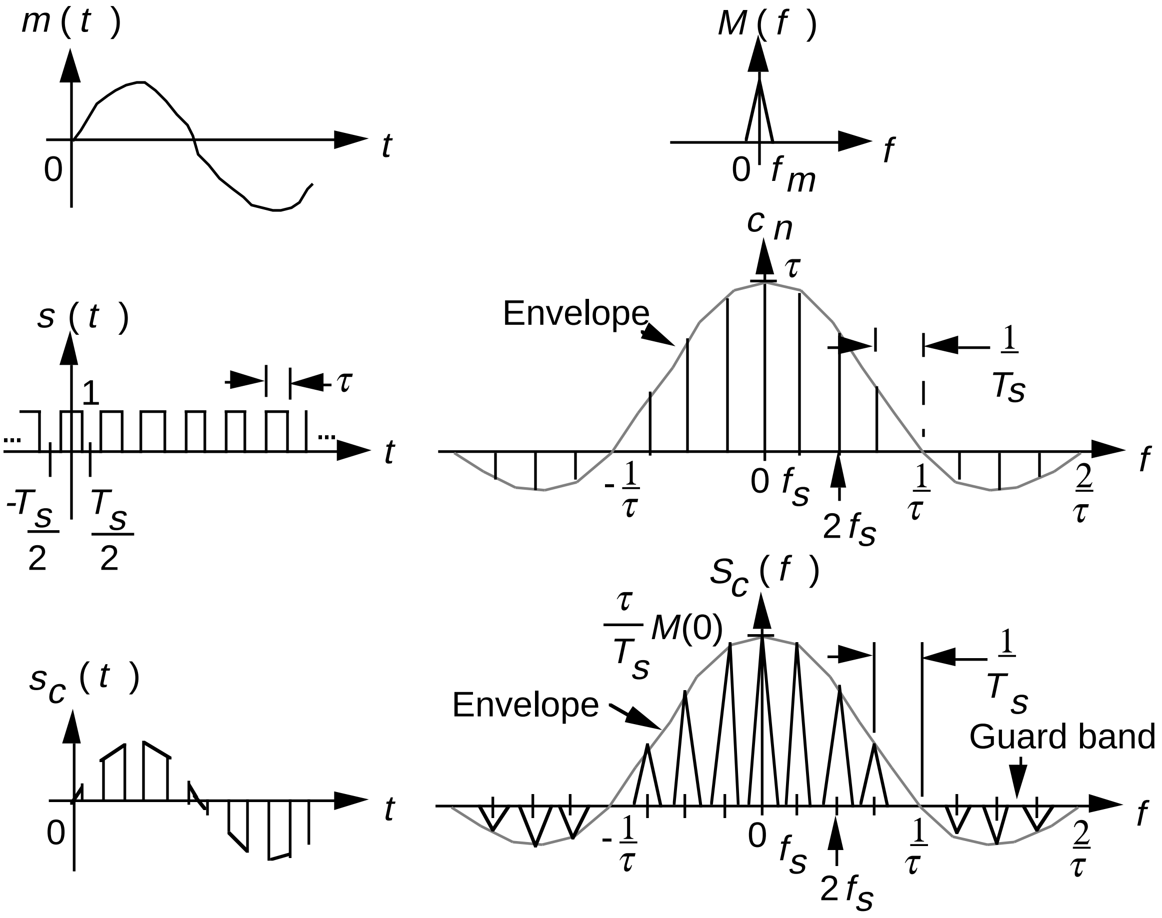


Figure 12.1 Waveform and spectra associated with signal sampling.

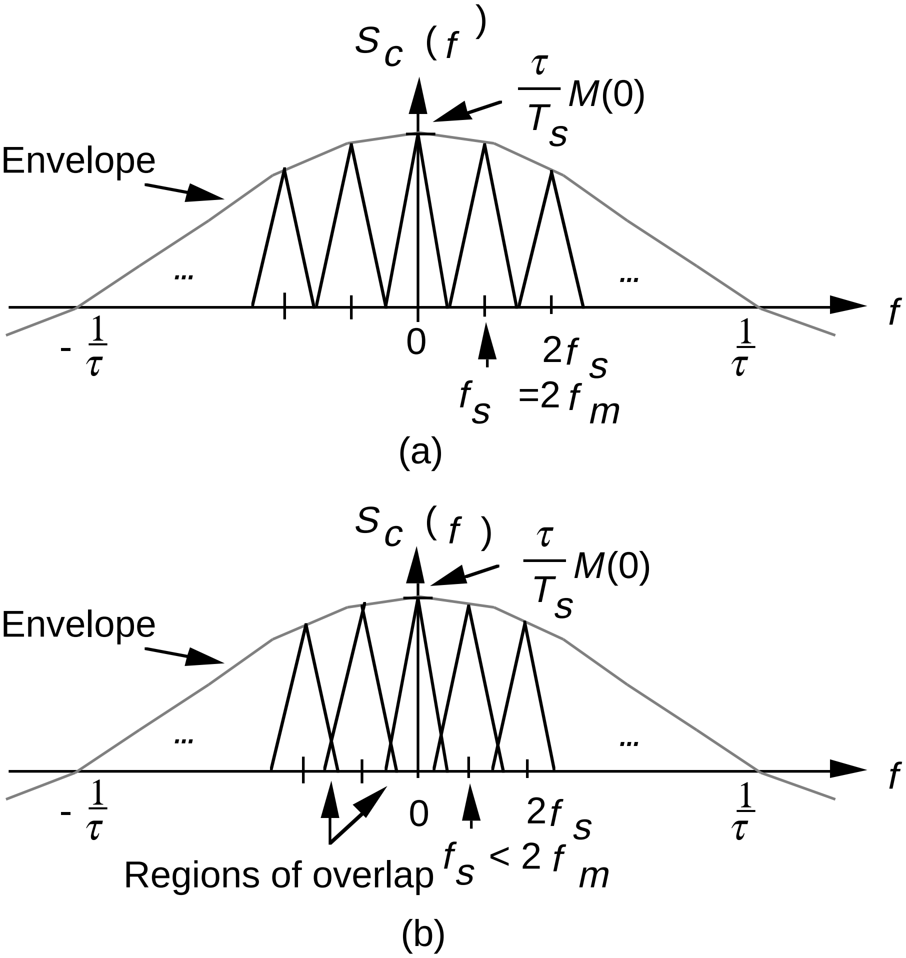


Figure 12.2 Signal spectra for (a) $f_s = 2f_m$ and (b) $f_s < 2f_m$.

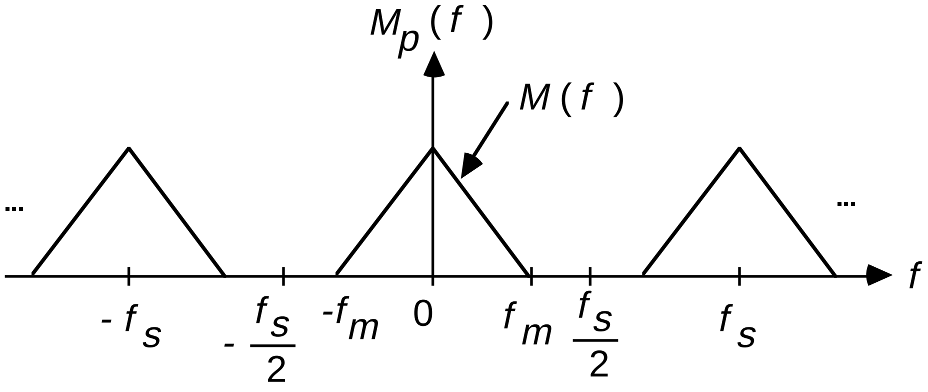


Figure 12.3 $M(f)$ used to make a periodic frequency function.

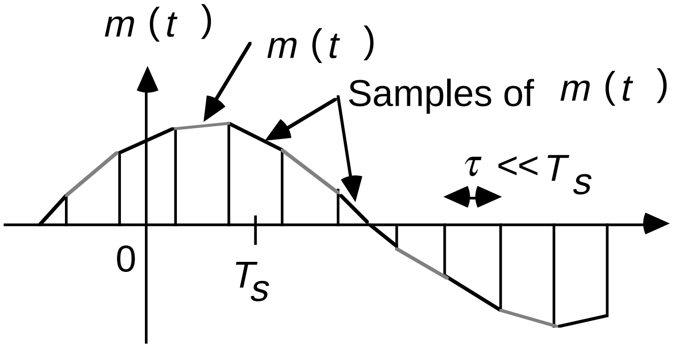


Figure 12.4 Samples of *m*(*t*).

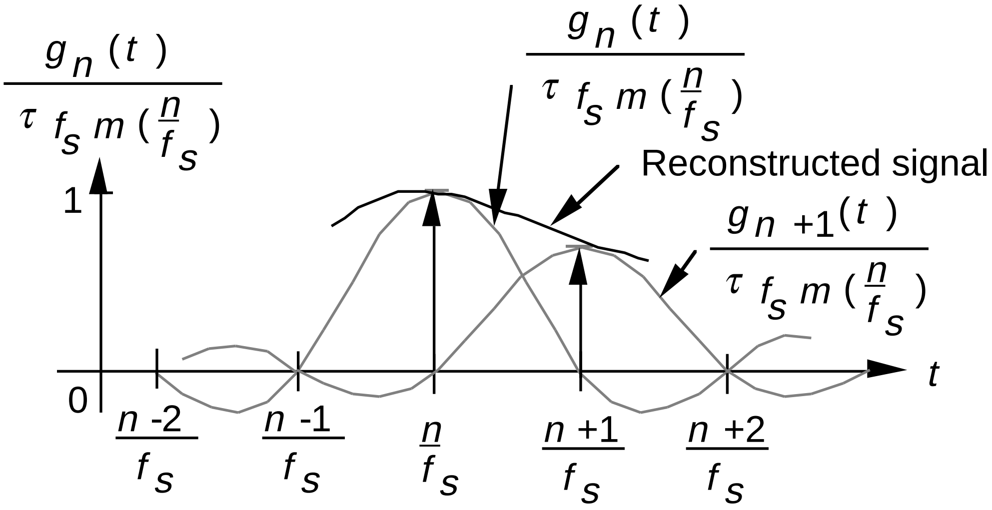


Figure 12.5 Filter response to input signal samples.

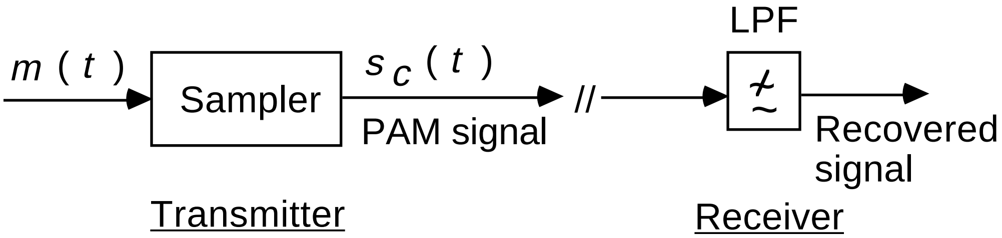
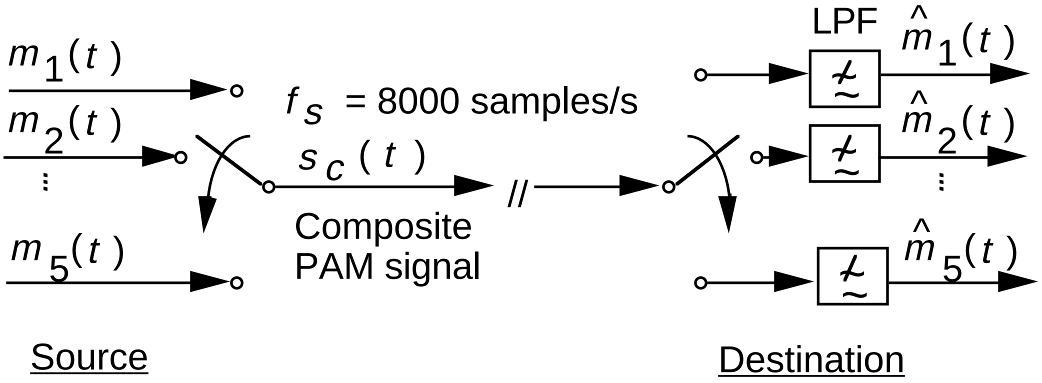
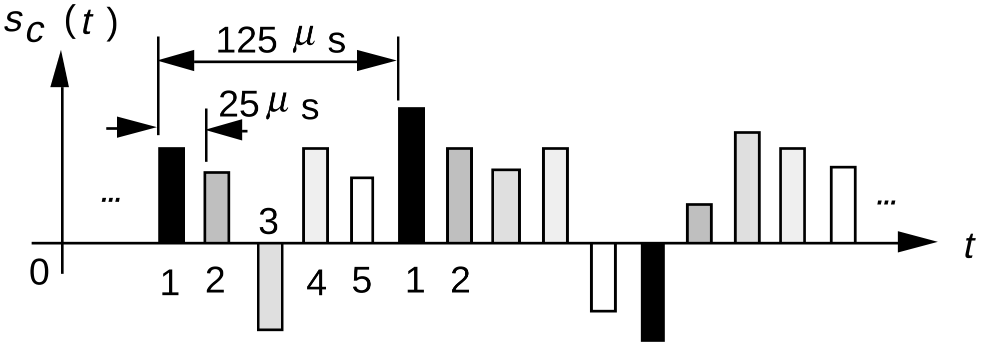


Figure 12.6 PAM system.



(a) Transmitter and receiver



(b) Waveform of TDM signal

Figure 12.7 TDM system.