16 Discrete-time Signals

A discrete-time signal is formally written as

$$s[k] \doteq f(kT) \qquad \forall \text{ integer } k \in (-\infty, \infty)$$

This means that s is defined as values of $f(t)$ only at $t = kT$. The signal is not defined at any other time.

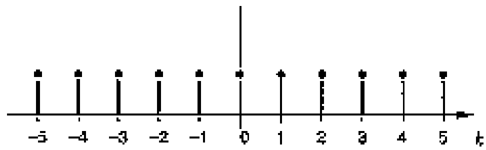
Essentially the signal is simply a list of values.

16.1 Basic Signals

Simple discrete-time signals that will be used later are:

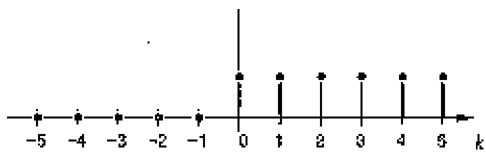
Unity Sequence

$$u[k] = 1$$



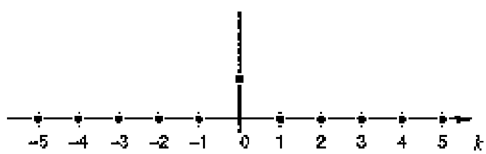
Unit-step Sequence

$$q[k] = \begin{cases} 1 & k \geq 0 \\ 0 & k < 0 \end{cases}$$



Impulse Sequence

$$\delta[k] = \begin{cases} 1 & k = 0 \\ 0 & k \neq 0 \end{cases}$$



16.2 Signal Manipulation

Discrete-time signals can be manipulated by a *transform* of the index variable of the signal

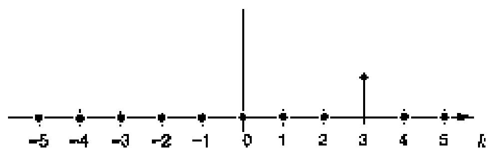
$$g[k] = f\left[\mathcal{G}(k)\right]$$

The index remains k. The transform $\mathcal{G}(k)$ must map integers to integers.

shifting

$$\delta[k-i] = \begin{cases} 1 & k = i \\ 0 & k \neq i \end{cases}$$

$\delta[k-3]$

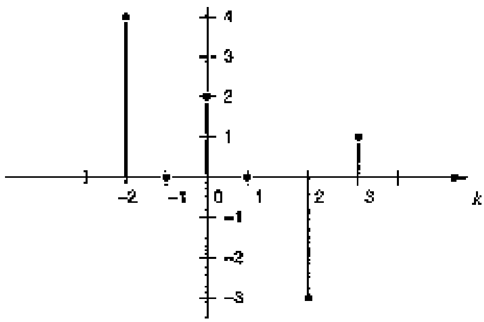


16.3 Sum of *impulse*functions

Any sequence can be considered as a sum of impulse sequences.

Considered, as an example, the sequence:

$$\begin{aligned} f[k] &= \{f[-2],\ f[-1],\ f[0],\ f[1],\ f[2],\ f[3]\} \\ &= \{4,\ 0,\ 2,\ 0,\ -3,\ 1\} \end{aligned}$$



Expressing this as a sum of impulse sequences gives:

$$f[k] = 4\delta[k+2] + 2\delta[k] - 3\delta[k-2] + \delta[k-3]$$

Basic Transform

In general, a sequence is constructed by a sum of impulse sequences:

$$F[k] = \sum_{j=-\infty}^{\infty} f[j]\delta[k-j]$$

This is called a *transform* from the j-domain to the k-domain.

Note the following:

- $f[j]$ is a function of j.
- $F[k]$ is a function of k (not j).
- This is a trivial *transform* because j and k are the same type of variable, and f and F are the same function.

The view of f as a function of index j, however, is changed to one viewed as a function of index k.

The utility of this manipulation comes from being able to modify the function in the second domain and returning it to the first. A transform pair gives a two-way path between the domains:

$$F[k] = \sum_{j=-\infty}^{\infty} f[j]\delta[k-j]$$

$$f[j] = \sum_{k=-\infty}^{\infty} F[k]\delta[j-k]$$

In this case the *base* functions are impulse sequences. Complex exponential functions (or sequences) are a much more useful base because they allow transforms from time and frequency domains. This idea will be dealt with later.

16.4 Sequence Construction

Basic sequences can be constructed as sums of impulse sequences.

Examples

Constant Sequence The constant unit sequence has a value of 1 for all k.

$$u[k] = \sum_{j=-\infty}^{\infty} \delta[k-j].$$

Rectangular Pulse The rectangular-pulse sequence is defined as:

$$P_N[k] = \begin{cases} 0 & k < -N \\ 1 & -N \leq k \leq N \\ 0 & k > N \end{cases}$$

so,

$$P_N[k] = \sum_{j=-N}^{N} \delta[k-j].$$

Sequences can also be constructed as samples of continuous-time signals.

Examples

Exponential Sequence The Exponential sequence is defined as:

$$f[k] = e^{akT}$$

which is a *sampling* of $f(t) = e^{at}$ with sampling period T.

The general exponential sequence is $f[k] = b^k \quad \forall$ integer k.

In this case $b = e^{aT}$.

Sinusoidal Sequence The Sinusoid sequence is defined as:

$$f[k] = \sin(\omega kT + \theta) \quad \forall \text{ integer } k.$$

16.5 Periodic Discrete-time Sequences

Periodic Discrete-time Signal

A discrete-time signal $f[k]$ is a periodic signal if there exits an integer N such that

$$f[k] = f[k+N] \quad \forall \text{ integer } k$$

If the signal is periodic then the period of the sequence is N. The smallest such N is the *fundamental period* of the sequence.

This definition implies that

$$f[k] = f[k+N] = f[k+2N] = f[k+3N] = \ldots = f[k+nN] \quad \forall \text{ integer } n$$

16.5.1 Periodic Sinusoidal Sequence

Consider the discrete-time signal $s[k]$ constructed by sampling the continuous time signal $\sin(\omega t)$ with a sampling period of T.

$$s[k] = \sin \omega k T$$

If $s[k]$ is periodic then $\exists$ integer N such that, $\forall$ integer k,

$$\begin{aligned} s[k] &= s[k+N] \\ \text{that is,} \qquad \sin \omega k T &= \sin \omega (k+N) T \\ &= \sin(\omega k T + \omega N T) \\ &= \sin \omega k T \cos \omega N T + \cos \omega k T \sin \omega N T. \end{aligned}$$

This is true if and only if $\cos \omega N T = 1$ and $\sin \omega N T = 0$. That is if and only if $\omega N T = 2\pi n$ for some integer n. This implies that if N exists then it is given by

$$N = \frac{2\pi n}{\omega T}.$$

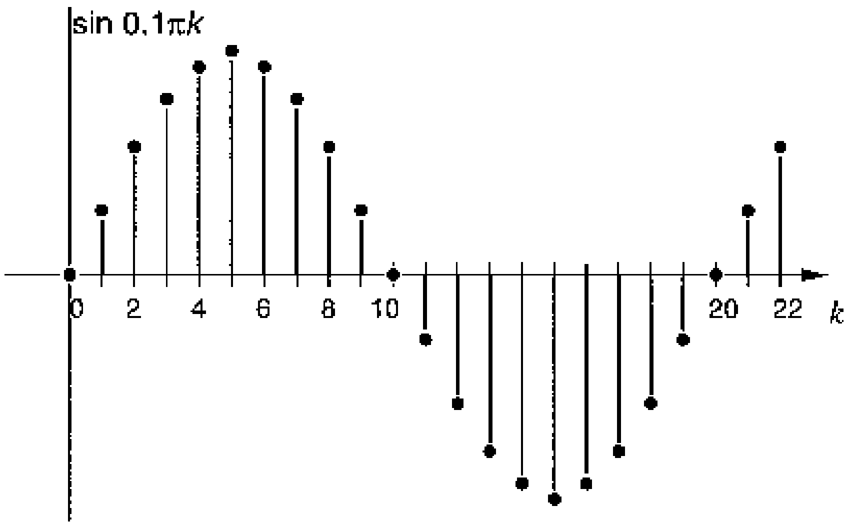
In conclusion, $s[k] = \sin \omega k T$ is periodic iff $\exists$ integer $n > 0$ such that $N = \frac{2\pi n}{\omega T}$ is a positive integer. If so, then N is the period of $s[k]$ and the smallest such N is the fundamental period.

Examples

Let $f[k] = \sin 0.1\pi k$. The period of f is given by:

$N = \frac{2\pi n}{0.1\pi} = 20n = 20, 40, 60, \ldots$ for $n = 1, 2, 3 \ldots$.

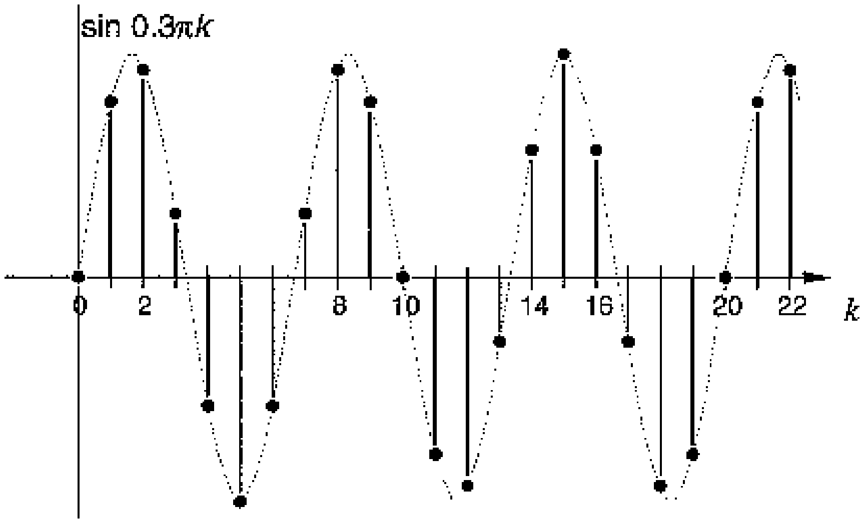
The fundamental period is 20.



Let $h[k] = \sin 0.3\pi k$. The period of h is given by:

$N = \frac{2\pi n}{0.3\pi} = \frac{20n}{3} = 20, 40, 60, \ldots$ for $n = 3, 6, 9 \ldots$

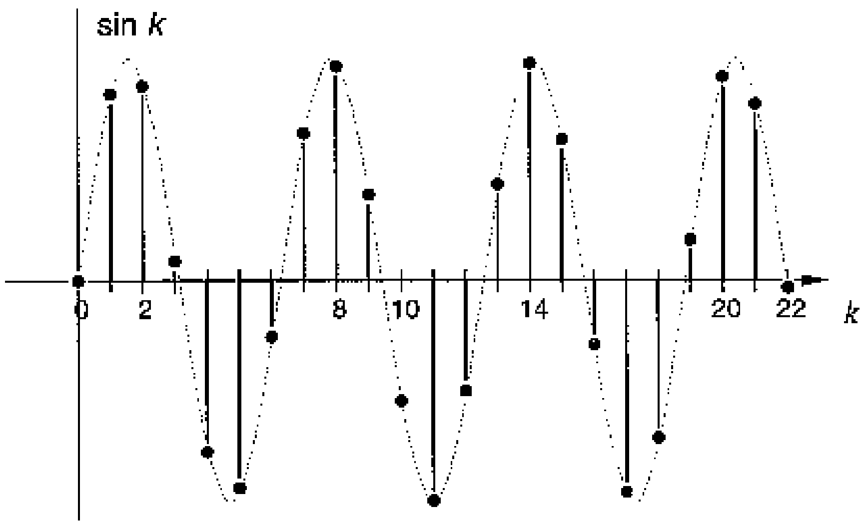
The fundamental period is 20.



Let $g[k] = \sin k$. The period of g, if it exists, is given by:

$$N = 2\pi n.$$

There does not exist an n such that $2n\pi$ is an integer. Thus $g[k]$ is not periodic.



16.6 Frequency

Discrete signals may be constructed by sampling a periodic continuous time signal. Not all discrete signals constructed this way are periodic as seen in the above example. It is reasonable, however, to expect that they must contain some aspect of the periodicity of the continuous signal from which they were derived. This expectation is the basis of the definition of the *frequency* of a discrete-time signal.

Before defining frequency, the process of sampling should be examined.

Sampling

> **Envelopes**
>
> A continuous-time signal $f(t)$ is an *envelope* of a discrete-time signal $g[k]$ if there exists a sampling period T such that
>
> $$g[k] = f(kT) \quad \forall\ k$$

If $\sin \bar{\omega} t$ is an envelope of $\sin \omega_o kT$ then so is $\sin\left(\bar{\omega} + \frac{2n\pi}{T}\right) t$.

This is because $\sin 2\pi nk = 0$ and $\cos 2\pi nk = 1$:

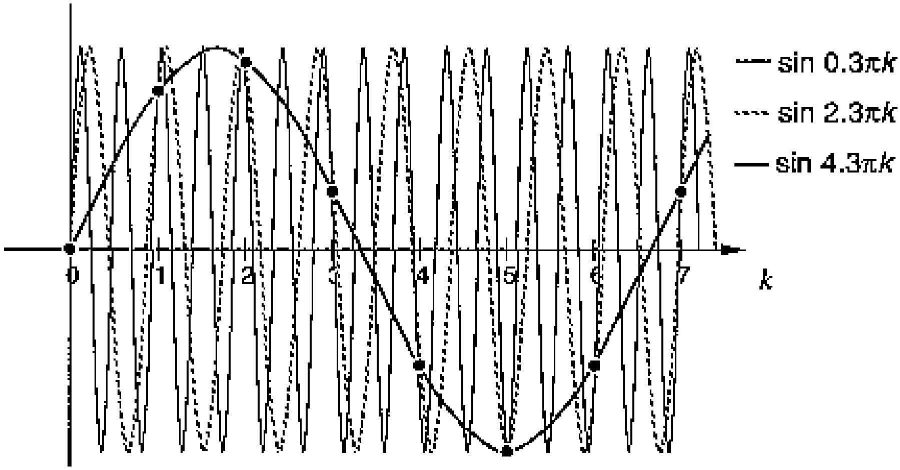
$$\begin{aligned}
\sin\left(\bar{\omega} + \frac{2n\pi}{T}\right) kT &= \sin(\bar{\omega} kT + 2n\pi k) \\
&= \sin \bar{\omega} kT \cos 2\pi nk + \cos \bar{\omega} kT \sin 2\pi nk \\
&= \sin \bar{\omega} kT \\
&= \sin \omega_o kT
\end{aligned}$$

Thus a sinusoidal sequence $\sin \omega_o kT$ has an infinite set of envelopes.

Example

Let $h[k] = \sin 0.3\pi k$. This signal can be constructed by sampling $h(t) = \sin 0.3\pi t$ with a sampling period $T = 1$.

In this case $\omega_o = 0.3\pi$. If $\bar{\omega} = \omega_o$, then $\sin(\bar{\omega})t$, $\sin(\bar{\omega} + 2\pi)t$, $\sin(\bar{\omega} + 4\pi)t$, ... are envelopes of $h[k]$.



16.6.1 Aliasing

A corollary of having an infinite set of envelopes is that there is an infinite number of sinusoids that all produce the same discrete time signal when sampled with a sampling period T.

Consider signals $f(t) = \sin \omega t$ and $g(t) = \sin\left(\omega + \frac{2n\pi}{T}\right)t$.

The sampling of $g(t)$ with sampling period T produces the same discrete time signal as the sampling of $f(t)$ because

$$\begin{aligned}\sin\left(\omega + \frac{2n\pi}{T}\right)kT &= \sin \omega kT \cos 2\pi nk + \cos \omega kT \sin 2\pi nk \\ &= \sin \omega kT\end{aligned}$$

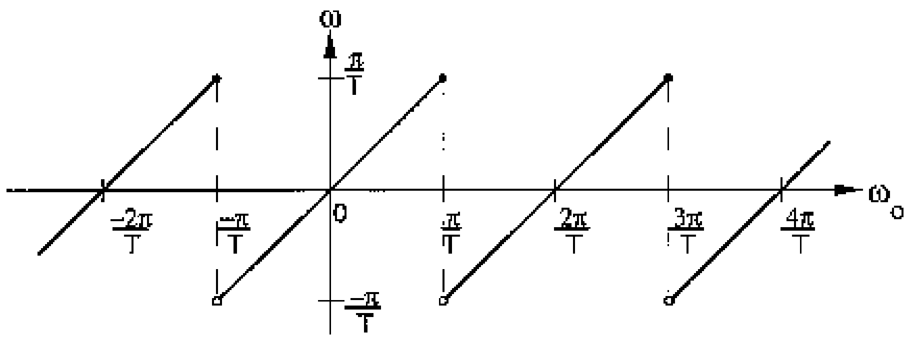
If the only information available about a continuous-time sinusoid is its discrete-time sampling, then it is not possible to uniquely determine its angular frequency. A sinusoid with angular frequency $\omega + \frac{2n\pi}{T}$ sampled with sampling period T will appear the same as one with angular frequency ω. When sampled, the former sinusoid is said to be aliased to — that is to have the same appearance as — the latter.

The frequency of a discrete-time signal is defined as being that of the alias with the smallest frequency:

Frequency of Discrete-time Signals

The frequency of a discrete-time signal $s[k] = \sin \omega_o kT$ is the frequency of the continuous-time signal $\sin \omega t$ with $-\pi < \omega T \leq \pi$ whose sampling with period T is $s[k]$

This definition implies that sampling a continuous-time signal with frequency $\omega_o \in (-\infty, \infty)$, at a sampling rate $1/T$, produces a discrete-time signal with a frequency $\omega \in (-\pi/T, \pi/T]$.



16.6.2 Sampling Theorem

If it is required that sinusoids in a range of frequencies are to be uniquely sampled then the maximum frequency ω_m must not exceed π/T. This frequency is called the *nyquist* frequency.

If the maximum frequency of a continuous-time signal is $\omega = 2\pi f$ then the sampling period required is $T \leq 1/2f$. That is, the sampling rate $R = 1/2T$ must be at least twice the maximum frequency.

The implication is the *sampling theorem*;

> A continuous-time signal with highest frequency component $\omega = 2\pi f$ can be completely recovered from a its discrete-time samples provided the sampling rate $R \geq 2f$. That is provided there are at least two samples per cycle.