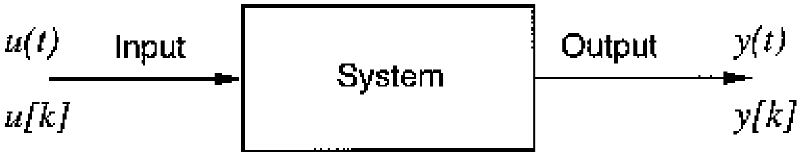
17 Systems

The term *system* is used here to name a machine, set of rules, or any collection of entities that produce a signal in response to other signals.



The system is a *black box* with an input and an output.

- The output signals are a response to the input signals.
- There can be any number of input and output signals.
- The input and output can be continuous-time signals or discrete-time signals or both.

This broad definition allows anything to be considered as a system. Building air conditioning is a system — temperature and humidity are inputs, and controlled air is an output. A doorbell is a system — the press of a button is the input, sound is the output. The internet is a system with an extremely large number of inputs and outputs. There are large systems such as the weather, the economy, and the telephone network. There are small systems such as radios, cd players and even cooking ovens.

It is useful to study the behaviour of systems using mathematical models. It has been shown that the signals that a system produces or responds to can be expressed as mathematical functions. The next step is to describe the system's response as a mathematical operation.

First it is necessary to classify systems, so that the appropriate mathematical model is used. Systems can be extremely complex but it is often possible to divide them into simpler components — similar to the way that complex signals can be represented as a sum of simpler signals.

The notation:

$$u_1(t); u_2(t); \ldots \rightarrow y_1(t); y_2(t); \ldots$$

indicates that the response of a system to input signals $u_1(t); u_2(t); \ldots$ is the output signals $y_1(t); y_2(t); \ldots$. The term t is usually time but may be any other type of independent variable.

The following definitions are given for a system with one input signal and one output signal.

17.1 Memory and Causality

The output at a particular time $t = t_o$ may depend on the value of an input at a past time $t < t_o$, and/or the present time $t = t_o$, and/or a future time $t > t_o$.

Memoryless systems depend only on present input. They do not need any knowledge of the value of signals at any other time.

A system has memory if its output $y(t_o)$ depends on the input $u(t)$ at a time $t \neq t_o$. The system has to *remember* the value that signals had or will have at times other than the present. This definition of memory implies that the system can see into the future (Not too hard when using mathematical models that tell you what the future is). In a practical application it is not possible to see into the future, however.

If the system depends on $u(t)$ for $t \leq t_o$ only then it is called a *causal* system. A memoryless system is therefore causal. The normal progression of time means that only causal systems, with time as the independent variable, can be built.

If the system depends on $u(t)$ for $t > t_o$ then it is *non-causal.*

It is useful to be able to study non-causal systems. The filters in modern cd audio players are based on non-causal designs. To implement these filters the output is calculated after a delay so that the future input is known.

17.2 Fundamental Systems

Systems can be constructed that implement the simple signal manipulations that were defined previously. Functions that shift, flip, multiply, modulate, or window signals can all be thought of as *systems*.

Unit time delay

The unit time delay system implements a shift operation. A continuous time unit delay system with input $u(t)$ has an output $y(t) = u(t-1)$. This is a delay of one second.

$$u(t) \;\rightarrow\; y(t) = u(t-1)$$

In a discrete-time unit delay system, the input signal is delayed by one index. The output is the previous input.

$$u[k] \;\rightarrow\; y[k] = u[k-1]$$

The unit time delay has memory and is causal.

17.3 Linearity

If a system produces the responses

$$u_1(t) \rightarrow y_1(t)$$

and

$$u_2(t) \rightarrow y_2(t)$$

then the system is linear if it is both *additive*, that is if

$$u_1(t) + u_2(t) \rightarrow y_1(t) + y_2(t)$$

and *homogeneous*, that is if

$$\alpha u_1(t) \rightarrow \alpha y_1(t)$$

where α is a constant.

Otherwise the system is nonlinear.

It is very useful to determine if a system is linear because its analysis can be simplified. It is important, however, to check the linearity of system before applying simplified analysis procedures.

Example:

The system:

$$y = mu$$

is linear. That is, if $y_1 = mu_1$ and $y_2 = mu_2$, then

$$u_1 + u_2 \rightarrow m(u_1 + u_2) = mu_2 + mu_1 = y_1 + y_2$$

and

$$\alpha u_1 \rightarrow m(\alpha u_1) = m\alpha u_1 = \alpha y_1$$

However, the system:

$$y = mu + c$$

is nonlinear. This is because it is not additive, if $y_1 = mu_1 + c$ and $y_2 = mu_2 + c$, then

$$u_1 + u_2 \rightarrow m(u_1 + u_2) + c = mu_2 + mu_1 + c = y_1 + y_2 - c$$

It is also not homogeneous,

$$\alpha u_1 \rightarrow m(\alpha u_1) + c = \alpha(mu_1 + c) + (1 - \alpha)c = \alpha y_1 + (1 - \alpha)c$$

Additivity and homogeneity together give the *principle of superposition*. This means that the response of a linear system to a complicated input can be calculated by summing the responses to simple inputs. The simple inputs would be those used to construct the complicated one.

17.4 Time Invariance

The response of a system to an input may depend on the time the input is applied.

For example, an automatic entry light system produces light in response to detected movement. Any time movement is detected the light turns on. This operation of the system is *time invariant.*

If the system is modified by adding a clock, so that no light is produced during daylight hours, then the system becomes *time variant.* The response to detected movement may be "do nothing" or "turn on the light," depending upon the time of day.

Formally, if

$$u(t) \rightarrow y(t)$$

and

$$u(t - t_o) \rightarrow y_o(t)$$

Then the system is time invariant if $y_o(t) = y(t - t_o) \qquad \forall t_o$.

17.5 Linear Time Invariant Systems

Normally the input to a system is not known for all time. Rather the input and output calculations begin at zero time. The initial starting conditions of the system may not be known.

Systems that are linear and time invariant possess the property that their output is the sum of:

- the response to the input assuming zero initial conditions, and
- the response to zero input with the actual initial conditions.

Linear Time Invariant (LTI) systems are easier to analyse because the initial conditions and response to an input can be separately considered. Useful results are obtained by simply ignoring the unknown initial conditions.

If the system is not LTI then the initial conditions cannot be readily ignored.

17.6 System State

Consider a causal system at time t_o. It will have an output that is the result of past inputs. Therefore, in order to determine the output, the past inputs must be known, possibly back to time equal to negative infinity. This can present an enormous calculation problem.

Alternatively, it may be easier to specify the effect of past input on future output. The minimum information necessary to specify this effect is all that is required, and this is usually much less than all past input information. This minimum information is called the *state* of the system at time t_o.

System State

> The state of a system, $x(t_o)$, at time $t = t_o$ is the information at t_o that together with the input $u(t)$, $t \geq t_o$, determines uniquely the system's output, $y(t)$, for all $t \geq t_o$.

Example:

The output of a system that performs integration, defined as

$$y(t) = \int_{-\infty}^{t} u(t)\,dt,$$

cannot be calculated without knowing the effect of $u(t)$ for all past t back to $t = -\infty$.

Rather than giving the input $u(t)$ for all $t < t_o$ and then requiring the calculation to be performed, it is easier to specify the state of the system at $t = t_o$. The state summarises the effect of past input on future output.

In this example, the state is simply a number that specifies the output at $t = t_o$.

$$\begin{aligned} y(t) &= \int_{-\infty}^{t_o} u(t)\,dt + \int_{t_o}^{t} u(t)\,dt \\ &= x(t_o) + \int_{t_o}^{t} u(t)\,dt \end{aligned}$$

The input to the system now consists of an *input/state* pair. That is, the state at $t = t_o$ and the input for all $t \geq t_o$. In general the output of any system is the result of an initial state and future input

$$x(t_o); u(t), t \geq t_o \;\rightarrow\; y(t)$$

17.7 Zero-State Response

If $y(t)$ is the output of a *linear time-invariant* (LTI) system then $y(t)$ for $t \geq t_o$ is the response to an initial state $x(t_o)$ and an input $u(t)$ for $t \geq t_o$.

This output of a linear time-invariant (LTI) system can always be decomposed as

total response = zero-state response + zero-input response

where the

- *zero-state response* is the output of the system when $x(t_o) = 0$, and the
- *zero-input response* is the output when $u(t) = 0$ for $t \geq t_o$.

This follows from linearity:

Because the system is linear, it possesses the additive property. That is the output response to the sum of two inputs is the sum of the response to each input. Thus if

$$x(t_o); u(t), t \geq t_o \ \rightarrow \ y(t)$$

then

$$0 + x(t_o); u(t) + 0, t \geq t_o \ \rightarrow \ y(t) = y_1(t) + y_2(t)$$

where

$$0; u(t), t \geq t_o \ \rightarrow \ y(t) = y_1(t)$$

and

$$x(t_o); 0, t \geq t_o \ \rightarrow \ y(t) = y_2(t)$$

In this case $y_1(t)$ is the zero-state response of the system and $y_2(t)$ is the zero-input response of the system.

The decomposition of the total response of a LTI system into zero-state and zero-input responses is useful because each can be analysed in isolation. An important zero-state response that portrays a lot of information about system is the response to an impulse.

17.8 Impulse Response

The impulse response of a system is defined as the output that occurs when the initial state is zero and the input is an impulse signal.

An impulse is defined by $\delta(t) = 0 \ \forall t \neq 0$ and $\int_{-\infty}^{\infty} \delta(t)dt = 1$.

Given an LTI system

$$x(t_o); u(t), t \geq t_o \ \rightarrow \ y(t)$$

then the zero-state impulse response is $h(t)$, $t \geq 0$ given by $h(t) = y(t)$ when $x(0) = x(t_o) = 0$ and $u(t) = \delta(t)$:

$$0; \delta(t), t \geq 0 \ \rightarrow \ h(t)$$

In simple terms, the impulse response of a system is the output that occurs when all inputs from $t = -\infty$ up to $t = t_o$ were held at zero (i.e. the output was at its zero-state) and then an impulse function is applied to the input.

Next it will be shown that the zero-state response of any LTI system to any input signal can be determined from its impulse response. The process of calculating the zero-state response of the system from its impulse response and the input signal is called *convolution*. This calculation will be developed firstly for discrete LTI systems and then repeated for continuous-time LTI systems.