18 Discrete Convolution

Consider a discrete system, $u[k] \rightarrow y[k]$, $k \geq 0$, that is

linear,

time-invariant, and

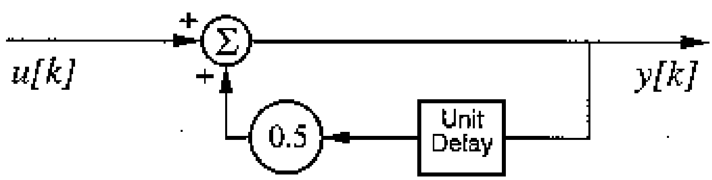
is at *zero-state* (i.e. the inputs were zero for all $k < 0$).

Its *zero-state impulse response* is $h[k] = y[k]$, $k \geq 0$ when the input $u[k]$ is an impulse signal, $\delta[k]$, and the initial state is the zero-state.

Recall that

$$\delta[k] = \begin{cases} 1 & k = 0 \\ 0 & k \neq 0 \end{cases}$$

Example:



In the system shown above, $y[k] = u[k] + 0.5y[k-1]$.

The *zero-state impulse response* is $h[k] = y[k]$ when

- the input $u[k] = \delta[k]$ and
- the inputs and outputs are at the zero state for $k < 0$.

Thus the zero state for this system is described by:

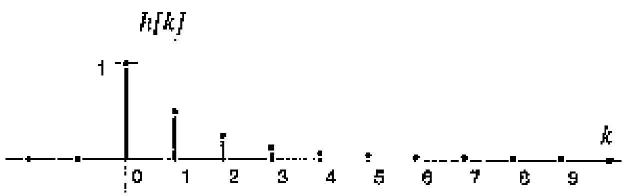
$y[-1] = 0$ and was established with $u[k] = 0 \ \forall k < 0$.

The impulse response can be determined at each k given that we know $y[-1] = 0$:

k	$y[k]$	$h[k]$
0	$\delta[0] + 0.5y[-1] = 1 + 0.5 \times 0$	1
1	$\delta[1] + 0.5y[0] = 0 + 0.5 \times 1$	1/2
2	$\delta[2] + 0.5y[1] = 0 + 0.5 \times 1/2$	1/4
3	$\delta[3] + 0.5y[2] = 0 + 0.5 \times 1/4$	1/8
4	$\delta[4] + 0.5y[3] = 0 + 0.5 \times 1/8$	1/16
4	$\delta[5] + 0.5y[4] = 0 + 0.5 \times 1/16$	1/32
.	.	.
.	.	.
.	.	.
n	$\delta[n] + 0.5y[n-1]$	$1/2^n$

The zero-state impulse response of this system $u[k] \rightarrow y[k]$ is therefore

$$h[k] = \frac{1}{2^k}.$$



18.1 Convolution

> **Proposition:** The zero-state response of a LTI system to any input can be determined from its zero-state impulse response.

That is, the response $y[k]$ due to input $u[k]$ can be determined from the response $h[k]$, $k \geq 0$ to an impulse input $\delta[k]$.

Discussion Any input sequence can be written as an infinite sum of inputs.

$$u[k] = \sum_{i=0}^{\infty} u[i]\delta[k-i],$$

which for any k is a sum of impulses multiplied by constants equal to $u[i]$. ($u[i]$ is not a function of k, so it is constant as far as $u[k]$ is concerned.)

The system is LTI, so it is time-invariant, homogeneous, and additive.

Time-invariance implies that the response is the same no matter when the input is applied, so if

$$\delta[k] \;\rightarrow\; h[k]$$

then

$$\delta[k-i] \;\rightarrow\; h[k-i]$$

Homogeneity implies that multiplying the input by a constant has the effect of multiplying the response by the same constant. Thus

$$u[i]\delta[k-i] \;\rightarrow\; u[i]h[k-i]$$

Additivity implies that the response to the total input, $u[k] \;\rightarrow\; y[k]$, is the sum of the responses to each $u[i]\delta[k-i]$ input. That is, if $u[i]\delta[k-i] \rightarrow u[i]h[k-i]$ then

$$y[k] = \sum_{i=0}^{\infty} u[i]h[k-i],$$

Note that $h[k] = 0 \; \forall k < 0$ because zero-state at $k = 0$ is assumed, so it follows that $h[k-i] = 0 \; \forall i > k$. Thus, there is no utility in summing beyond $i = k$, because the terms become zero.

The result is the *Discrete Convolution* of $u[k]$ and $h[k]$.

$$y[k] = \sum_{i=0}^{k} h[k-i]u[i],$$

It is interesting to note that because i is never greater than k, the kth output does not depend on inputs that occur after the kth one. That is, the system is causal. If this were not the case then it would be necessary for the impulse response to be non-zero for some $k < 0$.

18.1.1 General Discrete Convolution

The general convolution operator is denoted by an asterisk '$*$'.

Discrete Convolution Operator

The zero-state response of a LTI system to any input $u[k]$ can be determined from its zero-state impulse response $h[k]$ as:

$$h[k] * u[k] = \sum_{i=-\infty}^{\infty} h[k-i]u[i]$$

If the zero-state response is required then $u[i] = 0 \; \forall i < 0$, so the summation need only start at $i = 0$.

If the system is causal then $h[k-i] = 0 \; \forall i > k$, so the summation need not go beyond $i = k$.

Also,

$$\begin{aligned} h[k] * u[k] &= \sum_{i=-\infty}^{\infty} h[k-i]u[i] \\ &= \sum_{k-i=-\infty}^{\infty} h[k-(k-i)]u[k-i] \\ &= \sum_{i=-\infty}^{\infty} u[k-i]h[i] \\ &= u[k] * h[k] \end{aligned}$$

Example:

Consider the previous example that had a zero-state impulse response of

$$h[k] = \frac{1}{2^k}$$

If the input is a step function

$$q[k] = \begin{cases} 1 & k \geq 0 \\ 0 & k < 0 \end{cases}$$

then the output can be calculated if the fact that $u[k] \rightarrow y[k] = u[k] + 0.5y[k-1]$ and $y[-1] = 0$ are known:

k	$y[k]$	$h[k]$
0	$q[0] + 0.5y[-1] = 1 + 0.5 \times 0$	1
1	$q[1] + 0.5y[0] = 1 + 0.5 \times 1$	$3/2$
2	$q[2] + 0.5y[1] = 1 + 0.5 \times 3/2$	$7/4$
3	$q[3] + 0.5y[2] = 1 + 0.5 \times 7/4$	$15/8$
4	$q[4] + 0.5y[3] = 1 + 0.5 \times 15/8$	$31/16$
4	$q[5] + 0.5y[4] = 1 + 0.5 \times 31/16$	$63/32$
.	.	.
.	.	.
.	.	.
n	$q[n] + 0.5y[n-1]$	$(2^{n+1} - 1)/2^n$

The output can also be calculated without any knowledge about the system other than its zero-state impulse response (assuming it is LTI). The calculation is a discrete convolution of $h[k]$ with $q[k]$.

$$\begin{aligned} y[k] &= h[k] * q[k] \\ &= \sum_{i=-\infty}^{\infty} q[i]h[k-i] \\ &= \sum_{i=-\infty}^{k} q[i]\frac{1}{2^{k-i}} \qquad \text{(note } h[k-i]=0 \text{ if } i>k) \\ &= \sum_{i=0}^{k} \frac{1}{2^{k-i}} \qquad \text{(note } q[i]=0 \text{ if } i<0) \end{aligned}$$

for $k=0$: $y[0] = \sum_{i=0}^{0} \frac{1}{2^{0-i}} = 1$

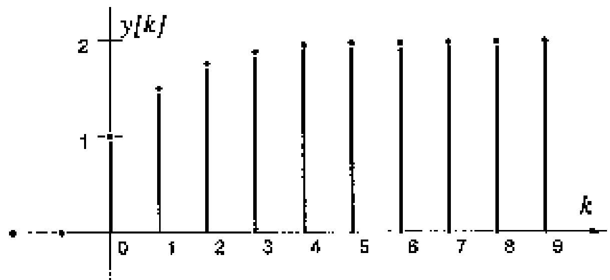
for $k=1$: $y[1] = \sum_{i=0}^{1} \frac{1}{2^{1-i}} = \frac{1}{2^{1-0}} + \frac{1}{2^{1-1}} = \frac{1}{2} + 1 = \frac{3}{2}$

for $k=2$: $y[2] = \sum_{i=0}^{2} \frac{1}{2^{2-i}} = \frac{1}{2^{2-0}} + \frac{1}{2^{2-1}} + \frac{1}{2^{2-2}} = \frac{1}{4} + \frac{1}{2} + 1 = \frac{7}{4}$

for $k=3$: $y[3] = \sum_{i=0}^{3} \frac{1}{2^{3-i}} = \frac{1}{2^{3-0}} + \frac{1}{2^{3-1}} + \frac{1}{2^{3-2}} + \frac{1}{2^{3-3}} = \frac{1}{8} + \frac{1}{4} + \frac{1}{2} + 1 = \frac{15}{8}$

for $k=n$: $y[n] = \sum_{i=0}^{n} \frac{1}{2^{n-i}} = \frac{1}{2^{n-0}} + \frac{1}{2^{n-1}} + \ldots + \frac{1}{2^{n-n}} = \frac{2^{n+1}-1}{2^n} = 2 - \frac{1}{2^n}$

Thus the response of the system $y[k] = u[k] + 0.5y[k-1]$ to a step signal is $q[k] \rightarrow y[k] = 2 - 2^{-k}$.



In this example the original form of the system response can be arrived at from the convolution integral:

$$\begin{aligned} y[k] &= \sum_{i=0}^{k} \frac{1}{2^{k-i}} \\ &= \frac{1}{2^{k-k}} + \sum_{i=0}^{k-1} \frac{1}{2^{k-i}} = 1 + \frac{1}{2}\sum_{i=0}^{k-1} \frac{1}{2^{k-1-i}} \\ &= 1 + 0.5y[k-1] \end{aligned}$$

18.2 Graphic Interpretation of Convolution

For each value of k the discrete convolution

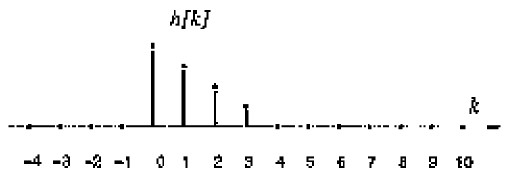
$$h[k] * u[k] = \sum_{i=-\infty}^{\infty} h[k-i]u[i]$$

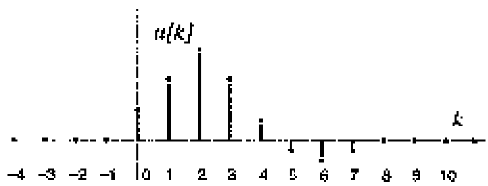
can be considered a succession of four operations.

1. h is flipped about $i = 0$. $(h[i] \mapsto h[-i])$
2. h is then shifted to the right by k. $(h[-i] \mapsto h[-(i-k)])$
3. h is then multiplied by u. $(h[k-i]u[i])$
4. The sequence is summed to give the kth value.

Example:

Consider the following impulse response and input signal.

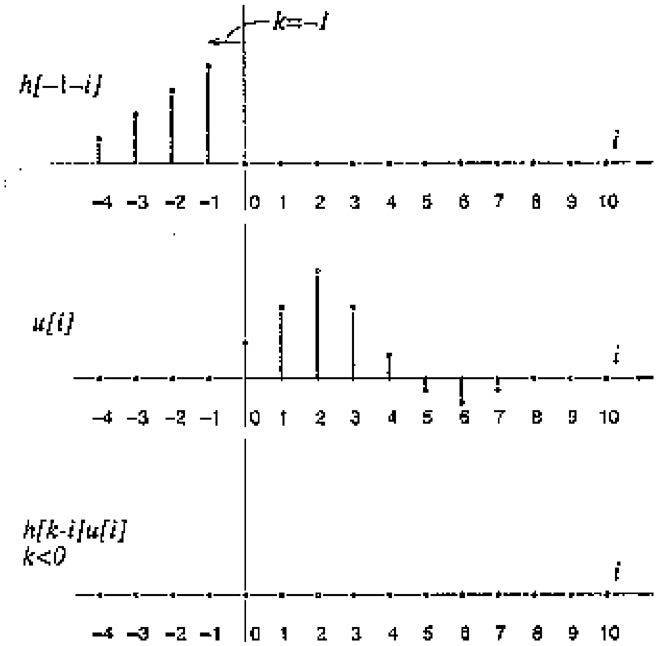




There are four non-zero terms in the impulse response and eight non-zero terms in the input signal.

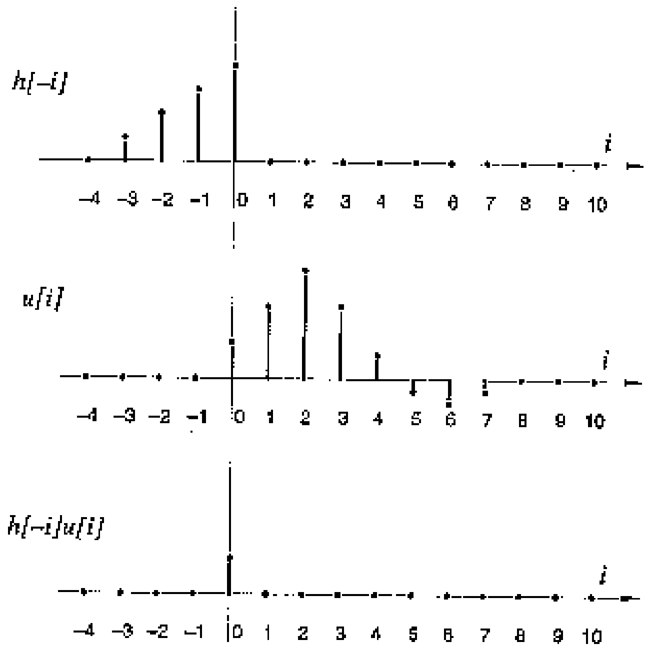
Consider a fixed value of k and look at the signals in terms of i.

For $k = -1$ the term $h[k-i]u[i]$ is zero, as would be the case for all i. The sum $\sum_{i=-\infty}^{\infty} h[k-i]u[i]$ is therefore zero.



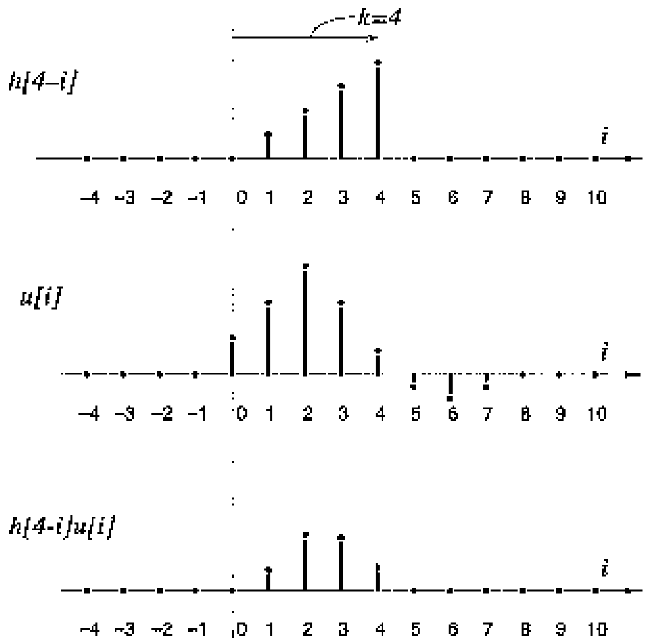
The sum $\sum_{i=-\infty}^{\infty} h[k-i]u[i]$ is zero for all $k < 0$.

For $k = 0$, $h[k-i]u[i]$ is not zero for all i because the signals overlap



The sum $\sum_{i=-\infty}^{\infty} h[k-i]u[i]$ is $h[0]u[0]$ for $k = 0$.

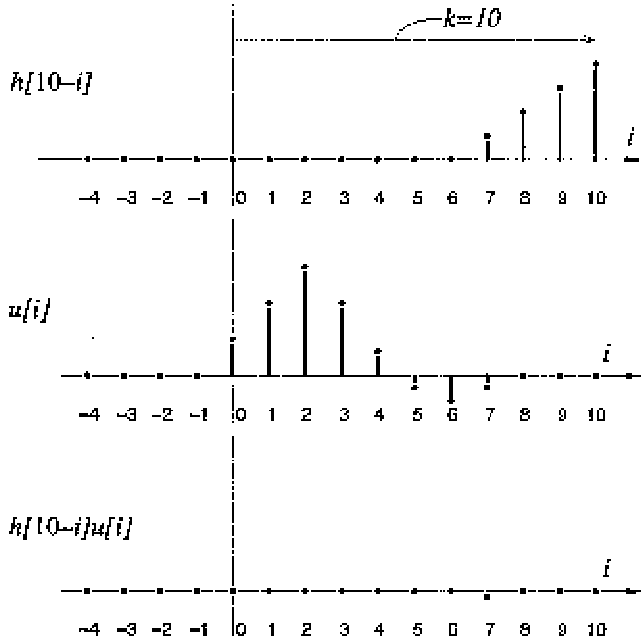
For k a little larger than zero, say $k = 4$ for example, $h[k-i]u[i]$ is again not zero because the signals overlap



The sum $\sum_{i=-\infty}^{\infty} h[k-i]u[i]$ for $k = 4$ is

$$h[3]u[1] + h[2]u[2] + h[1]u[3] + h[0]u[4]$$

For k equal to the sum of the lengths of the signals less two, $k = 4+8-2 = 10$ in this case, then $h[k-i]u[i]$ is still not zero for all i because the signals just overlap



The sum $\sum_{i=-\infty}^{\infty} h[k-i]u[i]$ for $k = 10$ is $h[3]u[7]$

For values of $k > 10$ the sum will be zero because the functions no-longer overlap.

It can be seen then that the convolution will be non-zero only if $0 \le k \le 10$, which is for 11 values of k.