19 FIR and IIR Systems

19.1 Sequence Length

The length of a sequence can be usefully defined as the length of the smallest contiguous portion of the sequence that has non-zero values.

The length of a sequence $u[k]$ is Q if there exists an i such that $u[k] = 0 \ \forall k < i$ and $u[k] = 0 \ \forall k \geq i + Q$. The smallest such Q is the length of the sequence.

In the previous example,

$h[k]$ has length 4 ($h[k] = 0 \ \forall k < 0$ and $h[k] = 0 \ \forall k \geq 4$),

$u[k]$ has length 8 ($u[k] = 0 \ \forall k < 0$ and $u[k] = 0 \ \forall k \geq 8$), and

$h[k] * u[k]$ has length 11 ($h[k] * u[k] = 0 \ \forall k < 0$ and $h[k] * u[k] = 0 \ \forall k \geq 11$), and

In general,

> If the length of the impulse is P and the length of the input sequence is Q then the length of their convolution will be $P + Q - 1$.

The fact that the length of the impulse response is significant is so much the case that systems are classified as either a:

> **Finite Impulse Response (FIR) System,**
> which has a finite number of non-zero entries in the impulse response, or an
>
> **Infinite Impulse Response (IIR) System,**
> which has an infinite number of non-zero entries in the impulse response.

19.1.1 FIR Systems

The output of an FIR system will eventually forget about a past input. The length of the output will be finite provided the input is finite. In either case, the effect of an input at k_o will eventually be zero at large enough k.

Example:

An example of an FIR system that is in common use is a moving average filter. A three term filter has an impulse response of

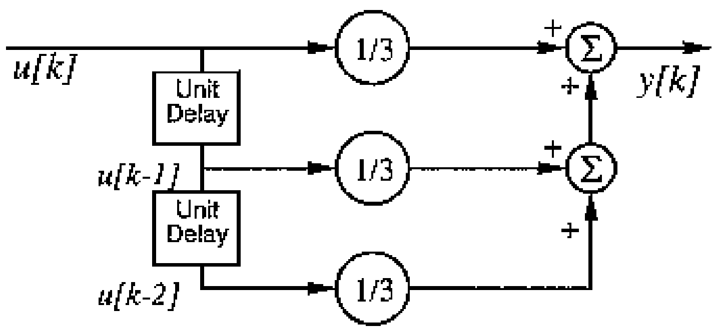
$$h[k] = \begin{cases} \frac{1}{3} & k = 0, 1, 2 \\ 0 & \text{otherwise} \end{cases}$$

The output is given by convolution as

$$\begin{aligned} y[k] &= u[k] * h[k] \\ &= \sum_{i=0}^{k} u[k-i]h[i] \\ &= h[0]u[k-0] + h[1]u[k-1] + h[2]u[k-2] + \ldots + +h[k]u[k-k] \\ &= \frac{1}{3}u[k] + \frac{1}{3}u[k-1] + \frac{1}{3}u[k-2] \\ &= \frac{u[k] + u[k-1] + u[k-2]}{3} \end{aligned}$$

The output at k is simply the average of the present and past two inputs. Inputs earlier than $k-2$ have no affect on the output.

It is possible to construct this system with a set of basic functions. Simple adders and multipliers are required. Delay functions are also required to remember the previous inputs.



19.1.2 IIR Systems

The output of an IIR system will show the affect of an input for ever. The output can be calculated as the convolution of the impulse response and the input. As seen above its length will be one less than the sum of the lengths of the impulse and the signal. Thus, no matter how short the input is, the output will be infinitely long if the impulse response is infinitely long.

Example:

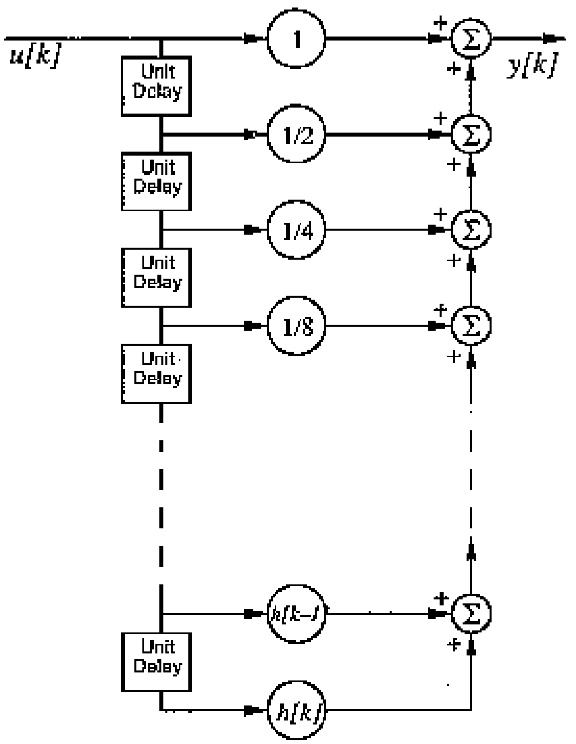
Consider a previous example, which has an infinite impulse response of

$$h[k] = \frac{1}{2^k}$$

The output is given by

$$\begin{aligned} y[k] &= u[k] * h[k] \\ &= \sum_{i=0}^{k} \frac{1}{2^i} u[k-i] \\ &= u[k] + \frac{1}{2}u[k-1] + \frac{1}{4}u[k-2] + \ldots + \frac{1}{2^k}u[0] \end{aligned}$$

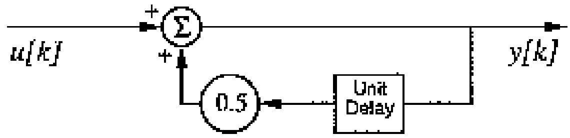
It is possible to think of constructing this system with a set of basic functions. The problem is that the system will have infinite size. It will need to be as large as the highest k for which the output is required.



The better alternative is to go back to the original description of the system, which was

$$y[k] = u[k] + 0.5y[k-1].$$

This is called a *difference equation* and can be thought of as relating the difference between outputs at differing k to the inputs at differing k. In this example the difference equation can be simply implemented.



19.2 Difference Equations

In a general discrete system the output can be written as

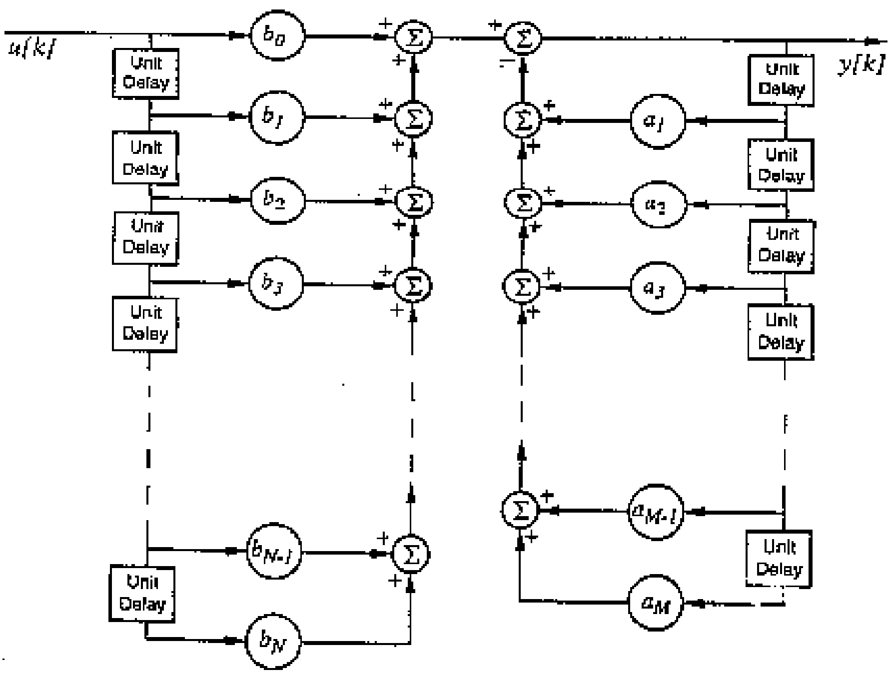
$$\begin{aligned} & y[k] + a_1 y[k-1] + a_2 y[k-2] + \ldots + a_M y[k-M] \\ & = b_0 u[k] + b_1 u[k-1] + b_2 u[k-2] + \ldots + b_N u[k-N] \end{aligned}$$

That is

$$y[k] + \sum_{i=1}^{M} a_i y[k-i] = \sum_{j=0}^{N} b_j u[k-j]$$

A system of this form can be constructed as

$$y[k] = \sum_{j=0}^{N} b_j u[k-j] - \sum_{i=1}^{M} a_i y[k-i]$$



The construction can be simplified if more is known about the system.

- If the system is has an FIR, then the coefficients a_i can be set to zero (i.e. M can be set to zero) in which case the coefficients b_j become the impulse response sequence ($b_j = h[j]$) and $N + 1$ is the length of the impulse response. The difference equation is then the convolution equation.

$$y[k] = \sum_{j=0}^{N} b_j u[k-j]$$

  Note that the calculation of the present output requires no knowledge of other outputs.

- If the system is has an IIR, then it may be easier to express it in terms of past outputs. That is make the a_i coefficients non-zero.

  The equation is then said to be *recursive* because it is necessary to calculate previous outputs before the present output can be calculated. A recursion back to the initial output is required.

  It is desirable to be able to determine this difference equation from the impulse response.

Example:

The convolution equation of the IIR system in the previous example can be manipulated to produce a simpler recursive equation:

Start with the convolution equation

$$y[k] = \sum_{j=0}^{k} \frac{1}{2^j} u[k-j]$$

A possible first step is to determine $y[k-1]$ and the isolate a term equal to $y[k]$:

To do this change k to $k-1$ and simplify:

$$\begin{aligned} y[k-1] &= \sum_{j=0}^{k-1} \frac{1}{2^j} u[k-1-j] \\ &= \sum_{j=1}^{k} \frac{1}{2^{j-1}} u[k-j] \\ &= 2\sum_{j=1}^{k} \frac{1}{2^j} u[k-j] \end{aligned}$$

This summation is almost the convolution equation that gives $y[k]$. A term can be added and subtracted to complete the convolution:

$$\begin{aligned} y[k-1] &= 2\sum_{j=1}^{k} \frac{1}{2^j} u[k-j] \\ &= +2\frac{1}{2^0} u[k-0] - 2\frac{1}{2^0} u[k-0] + 2\sum_{j=1}^{k} \frac{1}{2^j} u[k-j] \\ &= -2u[k] + 2\sum_{j=0}^{k} \frac{1}{2^j} u[k-j] \\ &= 2y[k] - 2u[k] \end{aligned}$$

This gives the expected difference equation

$$y[k] - 0.5y[k-1] = u[k]$$