20 Convolution of Continuous-time Signals

Convolution applied to continuous-time signals is similar to discrete convolution describe previously. The independent variable becomes t rather than k, and the summations become integrals.

Consider a discrete system, $u(t) \rightarrow y(t)$, $t \geq 0$, that is

linear,

time-invariant, and

is at *zero-state* (i.e. the inputs were zero for all $t < 0$).

Its *zero-state impulse response* is $h(t) = y(t)$, $t \geq 0$ when the input $u(t)$ is an impulse signal, $\delta(t)$.

20.1 Continuous-Time Impulse

The continuous-time impulse is defined by two conditions:

1. $\delta(t) = 0 \qquad \forall\, t \neq 0$, and
2. $\int_{-\infty}^{\infty} \delta(\tau)\, d\tau = 1$.

An impulse defined in this way has the following property:

For any continuous signal $y(t)$,

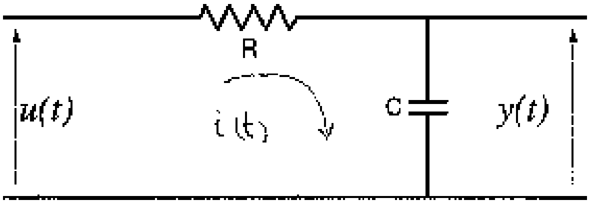
$$y(t)\delta(t - t_o) = y(t_o)\delta(t - t_o).$$

It follows that:

$$\begin{aligned}
\int_{-\infty}^{\infty} f(\tau)\delta(\tau - t_o)\, d\tau &= \int_{-\infty}^{\infty} f(t_o)\delta(\tau - t_o)\, d\tau \\
&= f(t_o) \int_{-\infty}^{\infty} \delta(\tau - t_o)\, d\tau \\
&= f(t_o)
\end{aligned}$$

Example:

Consider a low-pass filter as a system with input and output signal voltages $u(t)$ and $y(t)$.



$i(\tau) = \frac{u(\tau) - y(\tau)}{R} = C \frac{d}{d}$

$\int_{-\infty}^{t} \frac{u(\tau) - y(\tau)}{RC} \, d\tau = y$

[illegible]

In the system, $u(t) \;\rightarrow\; y(t) = \frac{1}{RC} \int_{-\infty}^{t} u(\tau) - y(\tau) \, d\tau$.

The *zero-state impulse response* is $h(t) = y(t) \; t \geq 0$ when

- the input $u(t) = \delta(t)$ and
- the inputs and outputs are at the zero state for $t < 0$.

Thus the zero state for this system is described by:

$y(0) = \int_{-\infty}^{0} y(\tau) \, d\tau = 0$ and was established with $u(t) = 0 \; \forall t < 0$.

The impulse response can be calculated as follows:

$$\begin{aligned}
y(t) &= \frac{1}{RC} \int_{-\infty}^{t} u(\tau) - y(\tau) \, d\tau \\
\text{so} \qquad h(t) &= \frac{1}{RC} \int_{-\infty}^{t} \delta(\tau) - h(\tau) \, d\tau \\
&= \frac{1}{RC} \int_{-\infty}^{t} \delta(\tau) \, d\tau - \frac{1}{RC} \int_{-\infty}^{0} h(\tau) \, d\tau - \frac{1}{RC} \int_{0}^{t} h(\tau) \, d\tau \\
&= \frac{1}{RC} - 0 - \frac{1}{RC} \int_{0}^{t} h(\tau) \, d\tau
\end{aligned}$$

Thus

$$\int_{0}^{t} h(\tau) \, d\tau = 1 - RC\,h(t),$$

for which the general solution is

$$h(t) = Ae^{Bt}$$

for constants A and B.

$$\begin{aligned}
1 - RC\,Ae^{Bt} &= \int_0^t Ae^{Bt}\,d\tau \\
&= \frac{A}{B}e^{Bt} - \frac{A}{B}e^{B0} \\
&= \frac{A}{B}e^{Bt} - \frac{A}{B} \\
\text{therefore } \frac{B}{A} - B\,RCe^{Bt} &= e^{Bt} - 1
\end{aligned}$$

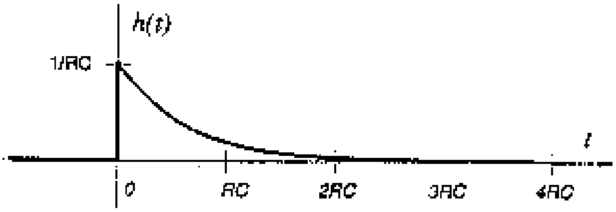
This implies that $B = -1/RC$ and $A = -B$.

Thus the *zero-state impulse response* of the low-pass filter system

$$u(t) \;\rightarrow\; y(t) = \frac{1}{RC}\int_{-\infty}^{t} u(\tau) - y(\tau)\,d\tau$$

is:

$$h(t) = \frac{1}{RC}e^{-t/RC} \qquad t > 0$$



20.2 Convolution

Proposition: The zero-state response of a LTI continuous-time system to any input can be determined from its zero-state impulse response.

That is, the response $y(t)$ due to input $u(t)$ can be determined from the response $h(t)$, $t \geq 0$ to an impulse input $\delta(t)$.

Discussion Any input sequence can be written as an integral function of the continuous-time impulse

$$u(t) = \int_{-\infty}^{\infty} u(\tau)\delta(t-\tau)\,d\tau.$$

For any t this is an integral of an impulse multiplied by a constants equal to $u(\tau)\,d\tau$. ($u(\tau)$ is not a function of t, so it is constant as far as $u(t)$ is concerned.)

The system is LTI, so it is time-invariant, homogeneous, and additive.

Time-invariance implies that the response is the same no matter when the input is applied, so if

$$\delta(t) \;\rightarrow\; h(t)$$

then

$$\delta(t-\tau) \;\rightarrow\; h(t-\tau)$$

Homogeneity implies that multiplying the input by a constant has the effect of multiplying the response by the same constant. Thus

$$u(\tau)\delta(t-\tau)\,d\tau \;\rightarrow\; u(\tau)h(t-\tau)\,d\tau$$

Additivity implies that the response to the total input, $u(t) \;\rightarrow\; y(t)$, is the sum of the responses to each $u(t)\delta(t-\tau)\,d\tau$ input. That is, if $u(\tau)\delta(t-\tau)\,d\tau \;\rightarrow\; u(\tau)h(t-\tau)\,d\tau$ then

$$y(t) = \int_{-\infty}^{\infty} u(\tau)h(t-\tau)\,d\tau.$$

Note that $h(t) = 0 \;\forall t < 0$ because zero-state is assumed, so it follows that $h(t-\tau) = 0 \;\forall \tau > t$. Therefore there is no utility in integrating beyond $\tau = t$, because the function becomes zero.

Also, if the input is zero for $t < 0$, then the integral can begin at $\tau = 0$.

The result is the *Convolution* of $u(t)$ and $h(t)$.

$$y(t) = \int_{0}^{t} h(t-\tau)u(\tau)\,d\tau.$$

Note that τ is never greater than t, so the output at t does not depend on inputs that occur after time t. That is, the system is causal. If this were not the case then it would be necessary for the impulse response to be non-zero for some $t < 0$.

20.2.1 General Convolution

The general convolution operator is also denoted by an asterisk '$*$'.

Convolution Operator

The zero-state response of a LTI system to any input $u(t)$ can be determined from its zero-state impulse response $h(t)$ as:

$$h(t) * u(t) = \int_{-\infty}^{\infty} h(t-\tau)u(\tau)\,d\tau$$

If the zero-state response is required then $u(\tau) = 0 \;\forall \tau < 0$, so the integral can start at $\tau = 0$.

If the system is causal then $h(t-\tau) = 0 \;\forall \tau > t$, so the integral need not go beyond $\tau = t$.

Also,

$$\begin{aligned} h(t) * u(t) &= \int_{-\infty}^{\infty} h(t-\tau)u(\tau)\,d\tau \\ &= \int_{\infty}^{-\infty} h(t-(t-\tau))u(t-\tau)\,d(t-\tau) \\ &= \int_{-\infty}^{\infty} u(t-\tau)h(\tau)\,d\tau \\ &= u(t) * h(t) \end{aligned}$$

Example:

Consider the previous filter example that has a zero-state impulse response of

$$h(t) = \frac{1}{RC} e^{-t/RC}$$

If the input is a step function

$$q(t) = \begin{cases} 1 & t > 0 \\ 0 & t < 0 \end{cases}$$

then the output can be calculated if the fact that $u(t) \rightarrow y(t) = \frac{1}{RC} \int_{-\infty}^{t} u(\tau) - y(\tau)\,d\tau$ and $y(0) = \frac{1}{RC} \int_{-\infty}^{0} y(\tau)\,d\tau = 0$ are known:

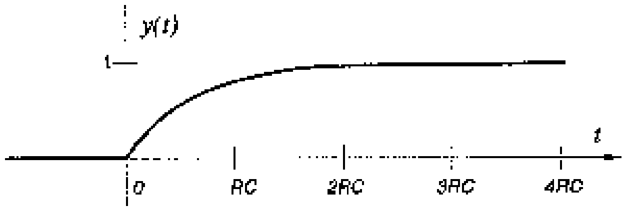
$$\begin{aligned} y(t) &= \frac{1}{RC} \int_{-\infty}^{t} u(\tau) - y(\tau)\,d\tau \\ &= \frac{1}{RC} \int_{-\infty}^{t} q(\tau) - y(\tau)\,d\tau \\ &= \frac{1}{RC} \int_{0}^{t} q(\tau)\,d\tau - \frac{1}{RC} \int_{-\infty}^{0} y(\tau)\,d\tau - \frac{1}{RC} \int_{0}^{t} y(\tau)\,d\tau \\ &= \frac{t}{RC} - \frac{1}{RC} \int_{0}^{t} y(\tau)\,d\tau \end{aligned}$$

To find the output is now necessary to solve this equation.

However, the output can be calculated without any knowledge about the system other than its zero-state impulse response (assuming it is LTI). The calculation is the convolution of $h(t)$ and $q(t)$.

$$
\begin{aligned}
y(t) &= q(t) * h(t) \\
&= \int_{-\infty}^{\infty} q(t-\tau)h(\tau)\,d\tau \\
&= \int_{0}^{\infty} q(t-\tau)\frac{1}{RC}e^{-\tau/RC}\,d\tau \qquad (\text{note } h(\tau) = 0 \text{ if } \tau < 0) \\
&= \int_{0}^{t} q(t-\tau)\frac{1}{RC}e^{-\tau/RC}\,d\tau \qquad (\text{note } q(t-\tau) = 0 \text{ if } \tau > t) \\
&= \int_{0}^{t} \frac{1}{RC}e^{-\tau/RC}\,d\tau \\
&= -e^{-t/RC} + e^{-0/RC} \\
&= 1 - e^{-t/RC}
\end{aligned}
$$

Thus the response of the RC filter to the step signal is $y(t) = 1 - e^{-t/RC}$.



This describes the charging of the capacitor to 1 volt over a time constant RC as expected.

This result can be checked with the integral equation derived above directly from the system response:

$$
\begin{aligned}
y(t) &= \frac{t}{RC} - \frac{1}{RC}\int_{0}^{t} y(\tau)\,d\tau \\
&= \frac{t}{RC} - \frac{1}{RC}\int_{0}^{t} 1 - e^{-\tau/RC}\,d\tau \\
&= \frac{t}{RC} - \frac{1}{RC}\int_{0}^{t} d\tau + \frac{1}{RC}\int_{0}^{t} e^{-\tau/RC}\,d\tau \\
&= \frac{1}{RC}\int_{0}^{t} e^{-\tau/RC}\,d\tau \\
&= y(t)
\end{aligned}
$$

20.3 Graphic Interpretation of Convolution

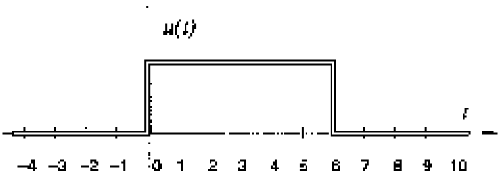
For each value of t the convolution

$$h(t) * u(t) \quad = \quad \int_{i=-\infty}^{\infty} h(k-i)u(\tau)\,d\tau$$

can be considered a succession of four operations.

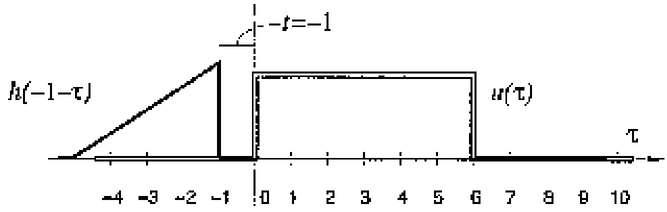
1. h is flipped. $(h(\tau) \mapsto h(-\tau))$
2. h is then shifted to the right by t. $(h(-\tau) \mapsto h(-(\tau - t)))$
3. h is then multiplied by u. $(h(t-\tau)u(\tau))$
4. The sequence is integrated (summed) to give the value at time t.

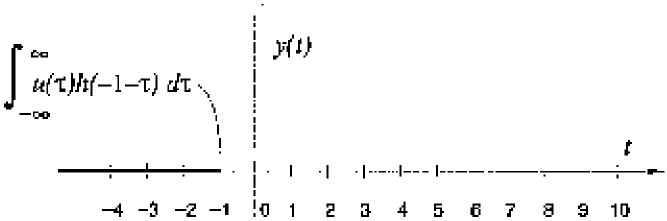
Example: Consider the following impulse response and input signal.



Consider a fixed value of t and look at the signals in terms of τ.

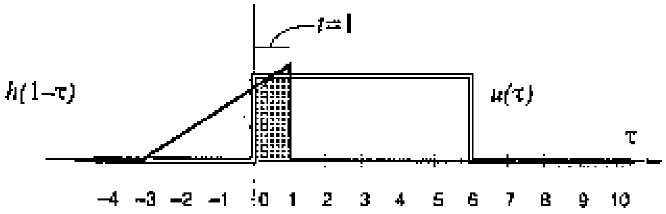
For $t = -1$ the product $h(t - \tau)u(\tau)$ is zero, as would be the case for all τ. The integration $\int_{-\infty}^{\infty} h(t - \tau)u(\tau)\, d\tau$ is therefore zero.

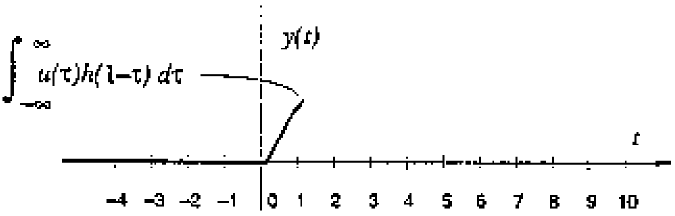




The integral $\int_{-\infty}^{\infty} h(t - \tau)u(\tau)\, d\tau$ is zero for all $t < 0$.

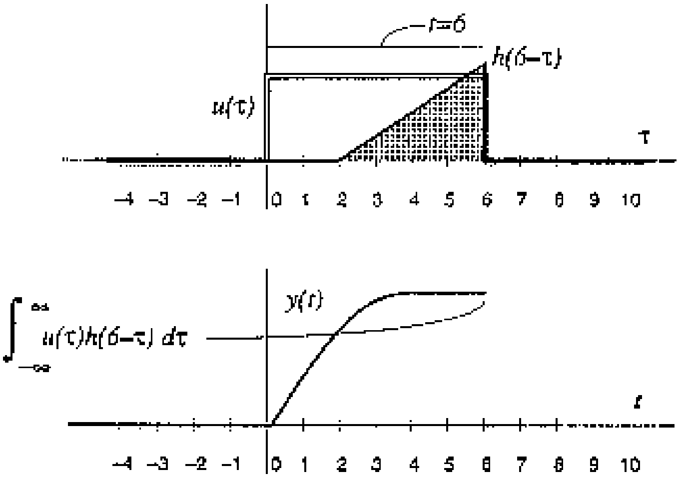
For $t = +1$, $h(t - \tau)u(\tau)$ is not zero for all τ because the signals overlap



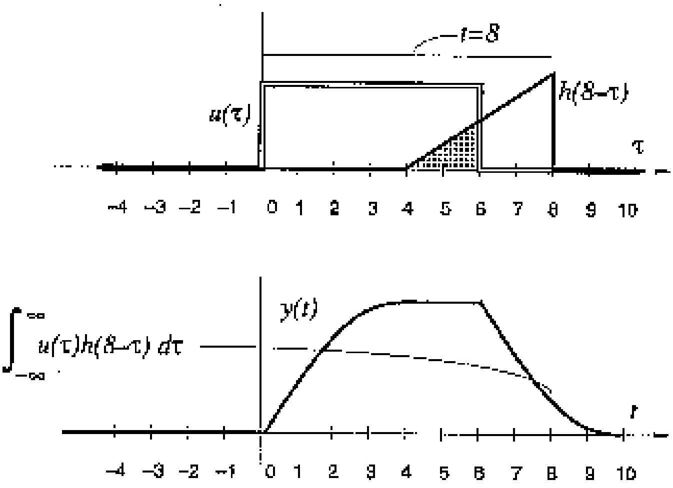


The integral $\int_{-\infty}^{\infty} h(t - \tau)u(\tau)\, d\tau = \int_{-\infty}^{\infty} h(1 - \tau)u(\tau)\, d\tau$ is not zero.

When $t = 6$ is reached, the integration includes the entire area under $h(t - \tau)$



As $t = 10$ is approached the amount by which the signals overlap is reduced, so the integral approaches zero.



20.4 Impulse versus Unit Step

It is not easy to produce an impulse signal in order to conduct an experiment. Its amplitude is undefined! An alternative signal that is easy to produce is the unit step. The impulse response can be measured if a relationship between the impulse and step responses can be established.

The definition of the unit step signal has thus far not been rigorous in regard to its value at $t = 0$. This is because it is not explicitly defined. Rather the formal definition of the *unit step* is in terms of a suitable relationship between the step and impulse signals.

20.4.1 Impulse and Unit Step Signals

The impulse $\delta(t)$ and step $q(t)$ functions are related by:

$$\frac{d}{dt} q(t) = \delta(t)$$

This definition of the impulse implies that the step is constant when $t \neq 0$:

$$\frac{d}{dt} q(t) = \delta(t) = 0 \qquad \forall t \neq 0$$

and the height of the step, $q(\infty) - q(-\infty)$, is unity:

$$\begin{aligned} 1 &= \int_{-\infty}^{\infty} \delta(\tau)\, d\tau \\ &= \int_{-\infty}^{\infty} \frac{d}{d\tau} q(\tau)\, d\tau \\ &= q(\tau)|_{-\infty}^{\infty} \\ &= q(\infty) - q(-\infty) \end{aligned}$$

20.4.2 Impulse and Unit Step Responses

There is also a similar relationship between an impulse response and a step response. The differential of a system's step response is the impulse response of the system.

That is, because the step response for a system with impulse response

$$\delta(t) \rightarrow h(t)$$

is

$$q(t) \rightarrow q(t) * h(t)$$

then

$$\frac{d}{dt}q(t) \rightarrow \frac{d}{dt}\left(q(t) * h(t)\right)$$

For this to be true it is necessary for $\frac{d}{dt}\left(q(t) * h(t)\right) = h(t)$.

$$\begin{aligned}
\frac{d}{d\tau}\left(q(t) * h(t)\right) &= \frac{d}{dt}\int_{-\infty}^{\infty} q(t-\tau)h(\tau)\,d\tau \\
&= \int_{-\infty}^{\infty} \frac{d}{dt}\left(q(t-\tau)h(\tau)\right)\,d\tau \\
&= \int_{-\infty}^{\infty} \left(\frac{d}{dt}q(t-\tau)\right)h(\tau)\,d\tau \\
&= \int_{-\infty}^{\infty} \delta(t-\tau)h(\tau)\,d\tau \\
&= h(t)
\end{aligned}$$

Example:

Recall the previous low-pass filter example with step response

$$y(t) = 1 - e^{-t/RC}.$$

The impulse response is the differential of the step response:

$$\begin{aligned}
h(t) &= \frac{d}{dt}\left(1 - e^{-t/RC}\right) \\
&= \frac{1}{RC}e^{-t/RC}
\end{aligned}$$

as expected.