# 21 Fourier Series

The following is an initial discussion of an analysis for representing a signal in terms of a set of periodic functions. The next section is a review of periodicity and properties of the periodic signals that will be used.

## 21.1 Periodic Signals

Recall the property of periodicity:

**Periodic Signal**

A signal $f(t)$ is a periodic signal with period $P$ if

$$f(t) = f(t+P) \quad \forall\, t \in (-\infty, \infty)$$

This implies that

$$f(t) = f(t+P) = f(t+2P) = f(t+3P) = \ldots$$

The smallest positive value of $P$ is the *fundamental period*.

The *fundamental frequency* is then $2\pi/P$ rad/s or $1/P$ Hz.

*Examples:*

$sin(\omega_o t + \theta)$ is periodic with period $2\pi/\omega_o$ s.

$f(t) = 1$ is periodic with any $P$. In this case the fundamental period is defined as zero.

## 21.2 Orthogonality of Complex Exponentials

A set of periodic functions, called complex exponentials, can be used as a set of *basis functions* for analysing any periodic signal. A property of this set that permits this is *orthogonality*.

Consider the complex exponential

$$e^{j\omega_o t} = \cos\omega_o t + j\sin\omega_o t.$$

A set of complex exponentials $\{\phi_m\}$ can be constructed where

$$\phi_m(t) = e^{jm\omega_o t} \quad m = 0,\ \pm 1,\ \pm 2, \ldots$$

For each $m$, $\phi_m$ is periodic with period $P_m = 2\pi/m\omega_o$. If $P = 2\pi/\omega_o$ then $P_m = P/m$. That is, the time interval $P$ consists of $m$ cycles of $\phi_m$.

### 21.2.1 Area under each period

**Proposition**

$$\int_{t_o}^{t_o+P} \phi_m(t)\,dt = \begin{cases} P & \text{if } m = 0 \\ 0 & \text{if } m \neq 0 \end{cases}$$

*Proof*

If $m = 0$ then

$$\begin{aligned} \int_{t_o}^{t_o+P} \phi_m(t)\,dt &= \int_{t_o}^{t_o+P} e^{j0\omega_o t}\,dt \\ &= \int_{t_o}^{t_o+P} dt \\ &= P \end{aligned}$$

If $m \neq 0$ then

$$\begin{aligned} \int_{t_o}^{t_o+P} \phi_m(t)\,dt &= \int_{t_o}^{t_o+P} e^{jm\omega_o t}\,dt \\ &= \frac{1}{jm\omega_o} e^{jm\omega_o t}\Big|_{t_o}^{t_o+P} \\ &= \frac{e^{jm\omega_o(t_o+P)} - e^{jm\omega_o t_o}}{jm\omega_o} \\ &= \frac{e^{jm\omega_o P} e^{jm\omega_o t_o} - e^{jm\omega_o t_o}}{jm\omega_o} \\ &= 0 \end{aligned}$$

because $e^{jm\omega_o P} = e^{jm\omega_o 2\pi/\omega_o} = e^{jm2\pi} = 1$.

This result confirms that there is a whole number of complex exponential cycles in each time interval $P$. The integration of the positive-half cycles is cancelled by the negative-half cycles.

### 21.2.2 Complex Conjugate

The complex conjugate of any complex number $z = a + jb$ is denoted by an asterisk as $z^* = a - jb$. Thus

$$\phi_m^*(t) = (e^{jm\omega_o t})^* = e^{-jm\omega_o t} = \phi_{-m}.$$

### 21.2.3 Property of Orthogonality

The orthogonality property of the set of complex exponentials $\{\phi_m\}$ is:

**Proposition**

If $\phi_k(t) = e^{jk\omega_o t}$ and $P = 2\pi/\omega_o$ then

$$\int_{t_o}^{t_o+P} \phi_m(t)\phi_n^*(t)\,dt = \begin{cases} P & \text{if } m = n \\ 0 & \text{if } m \neq n \end{cases}$$

*Proof*

Proof of this is easy after noting that $e^{jm\omega_o t}e^{-jn\omega_o t} = e^{j(m-n)\omega_o t}$ and thus

$$\phi_m(t)\phi_n^*(t) = \phi_{m-n}(t).$$

### 21.2.4 Complex Exponentials as Basis Functions

The set of complex exponentials $\{\phi_m\}$ are:

- periodic with the same period (although their *fundamental periods* are different) and are
- orthogonal to each other.

These two properties allow them to serve as a set of base functions for describing any periodic function with the same period. The description is called a *Fourier Series*.

## 21.3 Fourier Series of Periodic Functions

Consider a periodic function $f(t)$ with fundamental period $P$. Its fundamental frequency is $\omega_o = 2\pi/P$ rad/s.

It is possible to create a sequence of numbers from this function and a set of complex exponentials:

$$\begin{aligned} c_n &= \frac{1}{P}\int_{<P>} f(t)\phi_n^*(t)\,dt \\ &= \frac{1}{P}\int_{t_o}^{t_o+P} f(t)e^{-jn\omega_o t}\,dt \end{aligned}$$

If $f(t)$ and $\phi_n(t)$ are orthogonal then $c_n = 0$.

Otherwise, $c_n$ is a measure of the component of $f(t)$ that is *parallel* to $\phi_n(t)$. That is, the amount of $\phi_n(t)$ that is needed to contribute towards the construction of $f(t)$.

It seems then that $f(t)$ is a sum of complex exponentials with weights $c_n$.

To develop this idea, note that

$$c_m \frac{1}{P} \int_{<P>} \phi_m(t)\phi_n^*(t)\,dt = \begin{cases} c_m & \text{if } m = n \\ 0 & \text{if } m \neq n \end{cases}$$

It is therefore true that

$$\begin{aligned} c_n &= \sum_{m=-\infty}^{\infty} c_m \frac{1}{P} \int_{<P>} \phi_m(t)\phi_n^*(t)\,dt \\ &= \frac{1}{P} \sum_{m=-\infty}^{\infty} \int_{<P>} c_m\phi_m(t)\phi_n^*(t)\,dt \\ &= \frac{1}{P} \int_{<P>} \left( \sum_{m=-\infty}^{\infty} c_m\phi_m(t) \right) \phi_n^*(t)\,dt \end{aligned}$$

Since $c_n = \frac{1}{P}\int_{<P>} f(t)\phi_n^*\,dt$ it follows that

$$f(t) = \sum_{m=-\infty}^{\infty} c_m\phi_m(t)$$

This result is the *Fourier Series* representation of a periodic signal $f(t)$ with period $P$:

**Complex Exponential Fourier Series**

If $f(t)$ is periodic with period $P = 2\pi/\omega_o$ then

$$f(t) = \sum_{m=-\infty}^{\infty} c_m e^{jm\omega_o t}$$

$$c_m = \frac{1}{P} \int_{<P>} f(t) e^{-jm\omega_o t}\,dt$$

This pair of relationships establishes the *complex exponential Fourier series* (named in honour of the French mathematician Jean Baptiste Joseph Fourier, 1768-1830).

*Example:*

Consider the function:

$$f(t) = 1 - 3\cos 0.8\pi t + 2 sin 1.6\pi t + cos 2.8\pi t.$$

The fundamental frequency of this function is $\omega_o = 0.4\pi$, so the fundamental period is $P = 2\pi/0.4\pi = 5$.

In each fundamental period there is:

a zero frequency constant component,

2 cycles of a component with fundamental period $2\pi/0.8\pi = 5/2$,

4 cycles of a component with fundamental period $2\pi/1.6\pi = 5/4$, and

7 cycles of a component with fundamental period $2\pi/2.8\pi = 5/7$.

To represent $f(t)$ by a Fourier series, note that

$$\sin\omega t = \frac{e^{j\omega t} - e^{-j\omega t}}{2j} \quad \text{and} \quad \cos\omega t = \frac{e^{j\omega t} + e^{-j\omega t}}{2}$$

Thus

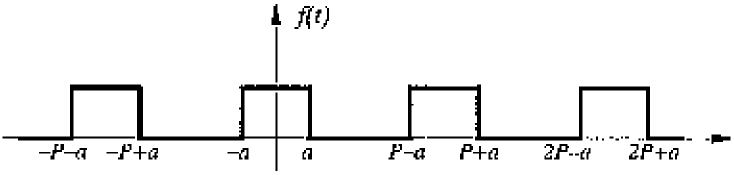
$$\begin{aligned} f(t) &= 1 - 3\cos 2\omega_o t + 2 sin 4\omega_o t + cos 7\omega_o t \\ &= e^0 - \frac{3}{2}\left(e^{j2\omega_o t} + e^{-j2\omega_o t}\right) \\ &\quad + \frac{2}{j2}\left(e^{j4w_o t} - e^{-j4w_o t}\right) + \frac{1}{2}\left(e^{j7w_o t} + e^{-j7w_o t}\right) \\ &= \phi_0 - \frac{3}{2}(\phi_2 + \phi_{-2}) + \frac{1}{j}(\phi_4 - \phi_{-4}) + \frac{1}{2}(\phi_7 + \phi_{-7}) \\ &= \frac{1}{2}\phi_{-7} + j\phi_{-4} - \frac{3}{2}\phi_{-2} + \phi_0 - \frac{3}{2}\phi_2 - j\phi_4 + \frac{1}{2}\phi_7 \end{aligned}$$

From inspection, the non-zero coefficients of the Fourier series $f(t) = \sum_{m=-\infty}^{\infty} c_m \phi_m(t)$ are:

$c_{-7} = 0.5$, $c_{-4} = j$, $c_{-2} = -1.5$, $c_0 = 1$, $c_2 = -1.5$, $c_4 = -j$, and $c_7 = 0.5$.

*Example:*

Consider the case of $f(t)$ being a train of rectangular pulses of amplitude 1, duration $2a$, and period $P$.



The fundamental frequency is $\omega_o = 2\pi/P$.

The Fourier coefficients of $f(t)$ are calculated as

$$\begin{aligned} c_m &= \frac{1}{P}\int\limits_{<P>} f(t)e^{-jm\omega_o t}\,dt \\ &= \frac{1}{P}\int\limits_{-a}^{a} e^{-jm\omega_o t}\,dt \end{aligned}$$

If $m = 0$ then

$$\begin{aligned} c_m &= \frac{1}{P}\int\limits_{-a}^{a} dt \\ &= \frac{2a}{P} \end{aligned}$$

if $m \neq 0$ then

$$\begin{aligned} c_m &= \frac{1}{P}\int\limits_{-a}^{a} e^{-jm\omega_o t}\,dt \\ &= \frac{1}{-jm\omega_o P}\left[e^{-jm\omega_o a} - e^{jm\omega_o a}\right] \\ &= \frac{-2j}{-jm\omega_o P}\left[\frac{e^{jm\omega_o a} - e^{-jm\omega_o a}}{2j}\right] \\ &= \frac{2}{m\omega_o P}\sin m\omega_o a \\ &= \frac{\sin 2\pi m(a/P)}{m\pi} \end{aligned}$$

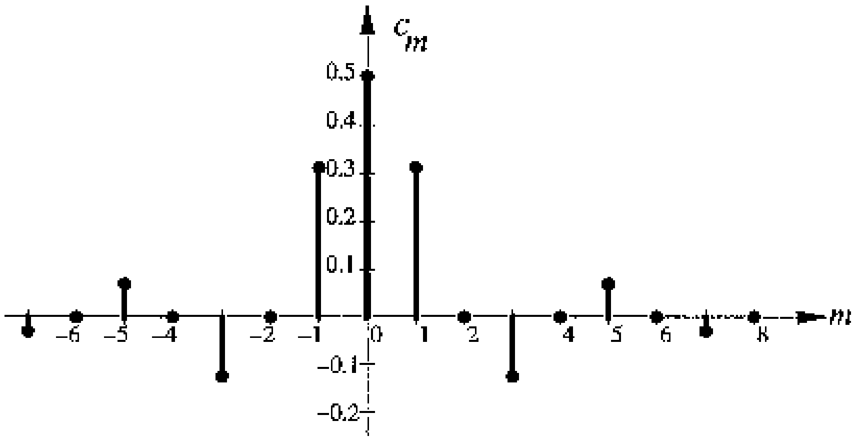
Thus:

$$c_m = \begin{cases} \frac{2a}{P} & \text{if } m = 0 \\ \frac{\sin 2\pi m(a/P)}{m\pi} & \text{if } m \neq 0 \end{cases}$$

Consider the case $a/P = 1/4$, i.e. a square wave function.

$$\begin{aligned} c_m &= \begin{cases} 0.5 & \text{if } m = 0 \\ \frac{\sin 0.5\pi m}{m\pi} & \text{if } m \neq 0 \end{cases} \\ &= \begin{cases} \frac{(-1)^{(m-1)/2}}{m\pi} & \text{if } m \text{ odd} \\ 0.5 & \text{if } m = 0 \\ 0 & \text{if } m \text{ even} \end{cases} \end{aligned}$$

A plot of $c_m$ looks like



Note that each $m$ corresponds to a complex exponential of frequency $m\omega_o$, where $\omega_o$ is the fundamental frequency of the signal. Thus the square-wave signal is the sum of a zero-frequency term and odd harmonics of its fundamental frequency.

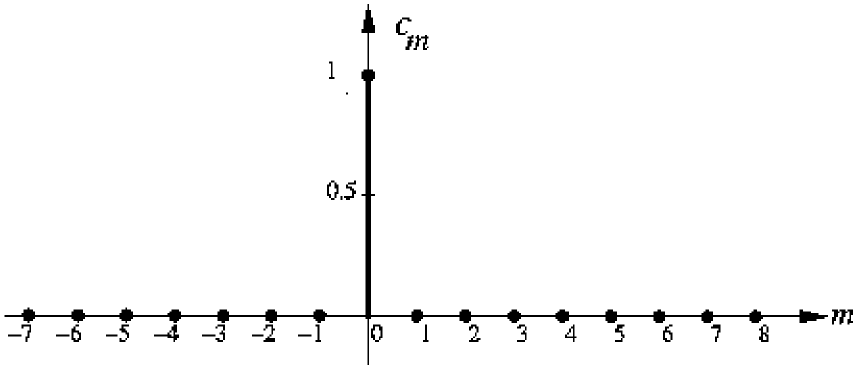
This can be expressed in terms of real sinusoids (rather than complex exponentials). That is, if $a/P = 1/4$, then

$$\begin{aligned}
y(t) &= \sum_{m=-\infty}^{\infty} c_m e^{jm\omega_o t} \\
&= \sum_{\substack{m=-\infty \\ m \text{ odd}}}^{-1} c_m e^{jm\omega_o t} + c_0 + \sum_{\substack{m=1 \\ m \text{ odd}}}^{\infty} c_m e^{jm\omega_o t} \\
&= c_0 + \sum_{\substack{m=1 \\ m \text{ odd}}}^{\infty} \left( c_m e^{jm\omega_o t} + c_{-m} e^{-jm\omega_o t} \right) \\
&= 0.5 + \sum_{\substack{m=1 \\ m \text{ odd}}}^{\infty} \frac{(-1)^{(m-1)/2}}{m\pi} \left( e^{jm\omega_o t} + e^{-jm\omega_o t} \right) \\
&= 0.5 + \sum_{\substack{m=1 \\ m \text{ odd}}}^{\infty} (-1)^{(m-1)/2} \frac{2}{m\pi} \cos m\omega_o t
\end{aligned}$$

Now consider the case $a/P = 1/2$, that is: $f(t) = 1$.

$$\begin{aligned} c_m &= \begin{cases} 1 & \text{if } m = 0 \\ \frac{\sin m\pi}{m\pi} & \text{if } m \neq 0 \end{cases} \\ &= \begin{cases} 1 & \text{if } m = 0 \\ 0 & \text{if } m \neq 0 \end{cases} \end{aligned}$$

A plot of $c_m$ looks like

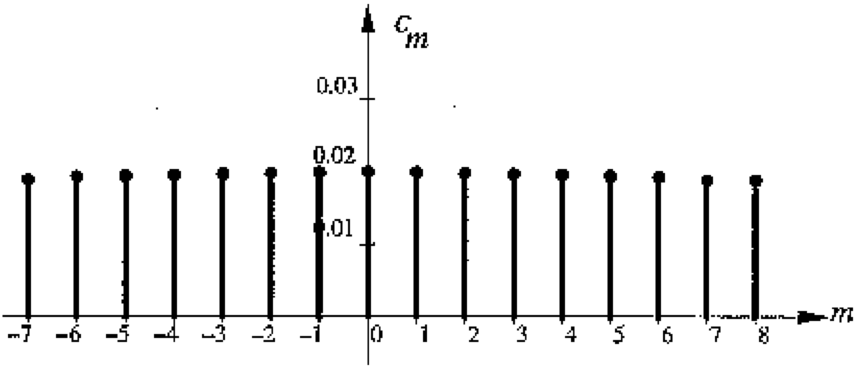


That is, the constant function contains only the zero-frequency component.

Finally consider the case $a/P = 1/100$, i.e. the pulses are very narrow.

$$c_m = \begin{cases} 1/50 & \text{if } m = 0 \\ \frac{\sin m\pi/50}{m\pi} & \text{if } m \neq 0 \end{cases}$$

A plot of $c_m$ looks like



This suggests that a narrow pulse contains nearly equal quantities of all frequencies. In fact the sum of all frequencies in equal quantity is a train of impulses:

$$\sum_{m=-\infty}^{\infty} \delta(t - mP) = \frac{1}{P} \sum_{m=-\infty}^{\infty} e^{jm\omega_0 t} \qquad \omega_0 = 2\pi/P$$