22 Fourier Analysis

22.1 Conditions for Existence of the Fourier Series

In order to generate a sequence of Fourier coefficients it is necessary to define a fundamental period and to integrate over that period. If this cannot be done for a particular signal then the Fourier series will not exist for that signal.

Formally, the conditions for the existence of a Fourier series for a signal $f(t)$, which is periodic with period P, are

1. $\int_{<P>} |f(t)|\, dt \leq M < \infty$, and
2. within one period, $f(t)$ has a finite number of discontinuities, maximas, and minimas.

These are called Dirichlet conditions (after German mathematician Peter Gustaw Lejeume Dirichlet, 1805-1857).

22.1.1 Gibb's Phenomena

If a Fourier series $F(t)$ is established from a function $f(t)$:

$$F(t) = \sum_{m=-\infty}^{\infty} c_m \phi_m(t)$$

where

$$c_m = \frac{1}{P} \int_{<P>} f(t)\phi_m^*(t)\, dt$$

then $F(t) = f(t)$ for almost all t, but not necessarily for all t.

There may be differences at discontinuities, but there is a finite number of these if the Dirichlet conditions are satisfied.

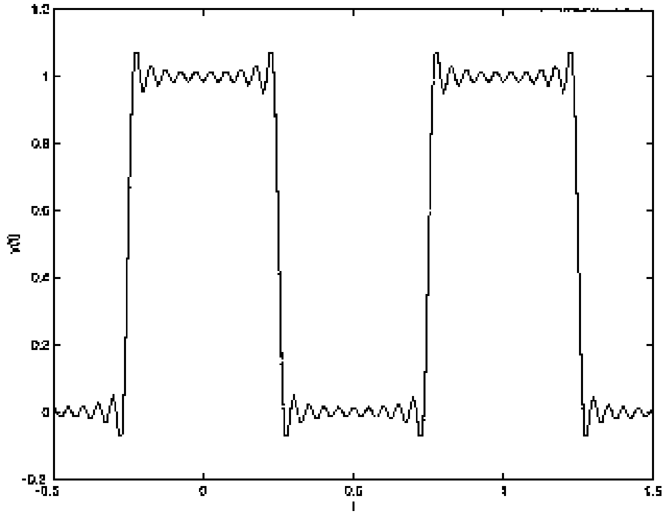
What will be true for all t is:

$$\int_{<P>} F(t)\phi_m^*(t)\, dt = \int_{<P>} f(t)\phi_m^*(t)\, dt$$

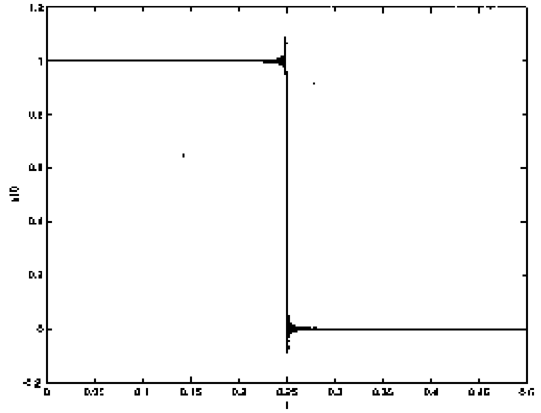
Example:

Consider the train of pulses example with $P = 1$ and $a = 0.25$.

The sum from $m = -20$ to $+20$ of its Fourier series looks like:



As the summation is continued the function maintains the 'ringing' shape, but it extends over an ever reducing time interval. The sum from $m = -500$ to $+500$ in the region of the step looks like:



In the infinite limit the result is constant between the discontinuities but has a positive and negative overshoot at the discontinuity. The magnitude of the overshoot is about 9% of the discontinuity.

This phenomena, called *Gibb's Phenomena*, occurs at every discontinuity. The reason for this is that a discontinuity cannot be constructed using continuous signals.

22.2 Trigonometric Fourier Series

Recall

$$e^{j\omega t} = \cos \omega t + j \sin \omega t$$

Using this, the Fourier series of $f(t)$ can be written as

$$\begin{aligned}
y(t) &= \sum_{m=-\infty}^{\infty} c_m e^{jm\omega_o t} \\
&= \sum_{m=-\infty}^{-1} c_m e^{jm\omega_o t} + c_0 + \sum_{m=1}^{\infty} c_m e^{jm\omega_o t} \\
&= c_0 + \sum_{m=1}^{\infty} \left(c_{-m} e^{-jm\omega_o t} + c_m e^{jm\omega_o t} \right) \\
&= c_0 + \sum_{m=1}^{\infty} \left(c_m \frac{e^{jm\omega_o t}}{2} + c_m \frac{e^{-jm\omega_o t}}{2} + j c_m \frac{e^{jm\omega_o t}}{2j} - j c_m \frac{e^{-jm\omega_o t}}{2j} \right. \\
&\qquad \left. + c_{-m} \frac{e^{jm\omega_o t}}{2} + c_{-m} \frac{e^{-jm\omega_o t}}{2} - j c_{-m} \frac{e^{jm\omega_o t}}{2j} + j c_{-m} \frac{e^{-jm\omega_o t}}{2j} \right) \\
&= c_0 + \sum_{m=1}^{\infty} \left(c_m \frac{e^{jm\omega_o t} + e^{jm\omega_o t}}{2} + j c_m \frac{e^{jm\omega_o t} - e^{-jm\omega_o t}}{2j} \right. \\
&\qquad \left. + c_{-m} \frac{e^{jm\omega_o t} + e^{-jm\omega_o t}}{2} - j c_{-m} \frac{e^{jm\omega_o t} - e^{-jm\omega_o t}}{2j} \right) \\
&= c_0 + \sum_{m=1}^{\infty} \left((c_m + c_{-m}) \frac{e^{jm\omega_o t} + e^{-jm\omega_o t}}{2} \right. \\
&\qquad \left. + j(c_m - c_{-m}) \frac{e^{jm\omega_o t} - e^{-jm\omega_o t}}{2j} \right) \\
&= c_0 + \sum_{m=1}^{\infty} \left((c_{-m} + c_m) \cos m\omega_o t + j(c_m - c_{-m}) \sin m\omega_o t \right)
\end{aligned}$$

By defining $a_0 = c_0$, $a_m = c_m + c_{-m}$, and $b_m = j(c_m - c_{-m})$, the Fourier series can be written as the *Trigonometric Fourier Series*

Trigonometric Fourier Series

If $y(t)$ is periodic with period $P = 2\pi/\omega_o$ then

$$y(t) = a_o + \sum_{m=1}^{\infty} a_m \cos m\omega_o t + \sum_{m=1}^{\infty} b_m \sin m\omega_o t$$

$$a_0 = \frac{1}{P} \int_{<P>} f(t)\, dt$$

$$a_m = \frac{2}{P} \int_{<P>} f(t) \cos m\omega_o t \, dt$$

$$b_m = \frac{2}{P} \int_{<P>} f(t) \sin m\omega_o t \, dt$$

22.3 Discrete Frequency Spectrum

Consider the Fourier series of a periodic signal $f(t)$

$$y(t) = \sum_{m=-\infty}^{\infty} c_m e^{jm\omega_o t}$$

Given that ω_o is known, the sequence $\{c_m\}$ is uniquely determined by $f(t)$.

Therefore there is a one-to-one correspondence between $f(t)$ and $\{c_m\}$.

All the information in $f(t)$ is also contained in the sequence $\{c_m\}$, which is called the *discrete frequency spectrum* of $f(t)$.

22.3.1 Simple properties of real periodic signals

If $f(t)$ is real then $f^*(t) = f(t)$ and the discrete frequency spectrum is complex symmetric, i.e. $c_m^* = c_{-m}$.

This is easy to show:

$$\begin{aligned} c_m^* &= \left(\frac{1}{P} \int_{<P>} f(t) e^{-jm\omega_o t}\, dt \right)^* \\ &= \frac{1}{P} \int_{<P>} f^*(t) e^{jm\omega_o t}\, dt \\ &= \frac{1}{P} \int_{<P>} f(t) e^{-j(-m)\omega_o t}\, dt \qquad\qquad f^*(t) = f(t) \\ &= c_{-m} \end{aligned}$$

It follows that if $c_m = \alpha_m + j\beta_m$ then

$$\alpha_m - j\beta_m = \alpha_{-m} + j\beta_{-m}$$

because $c_m^* = c_{-m}$.

Thus $\alpha_m = \alpha_{-m}$ and $-\beta_m = \beta_{-m}$.

The *amplitude spectrum* $|c_m|$ is an even function of m, and

the *phase spectrum* $\angle c_m$ is an odd function of m.

22.4 Parseval's Formula

The average power (energy per unit time) in a signal is defined as

$$P_{av} = \frac{1}{P} \int\limits_{<P>} f(t) f^*(t)\, dt$$

This information also exists in the signal's Fourier series.

$$\begin{aligned}
P_{av} &= \frac{1}{P} \int\limits_{<P>} f(t) f^*(t)\, dt \\
&= \frac{1}{P} \int\limits_{<P>} \left(\sum_{m=-\infty}^{\infty} c_m e^{jm\omega_o t} \right) f^*(t)\, dt \\
&= \sum_{m=-\infty}^{\infty} \left(\frac{1}{P} \int\limits_{<P>} c_m e^{jm\omega_o t} f^*(t)\, dt \right) \\
&= \sum_{m=-\infty}^{\infty} c_m \left(\frac{1}{P} \int\limits_{<P>} f(t) e^{-jm\omega_o t}\, dt \right)^* \\
&= \sum_{m=-\infty}^{\infty} c_m c_m^*
\end{aligned}$$

This is called Parsevals Formula for the average power of a signal

Parsevals Formula

If $f(t)$ is periodic with period $P = 2\pi/\omega_o$ then the average power of the signal $f(t)$ is

$$\begin{aligned}
P_{av} &= \sum_{m=-\infty}^{\infty} c_m c_m^* \\
&= \sum_{m=-\infty}^{\infty} |c_m|^2
\end{aligned}$$

where $c_m = \frac{1}{P} \int_{<P>} f(t) e^{-jm\omega_o t}\, dt$

Example:

Consider again the train of pulses example with $P = 1$ and $a = 0.25$. The power in the signal is

$$\begin{aligned} P_{av} &= \frac{1}{P} \int_{<P>} f(t) f^*(t)\, dt \\ &= \int_{-a}^{a} dt \\ &= 0.5 \end{aligned}$$

The Fourier series for the train of pulse signal is given by:

$$c_m = \begin{cases} \frac{(-1)^{(m-1)/2}}{m\pi} & \text{if } m \text{ odd} \\ 0.5 & \text{if } m = 0 \\ 0 & \text{if } m \text{ even} \end{cases}$$

The non-zero elements of this Fourier series are $c_0 = 0.5$, $c_{\pm 1} = \pm 1/\pi$, $c_{\pm 3} = \pm 1/3\pi$, $c_{\pm 5} = \pm 1/5\pi$, $c_{\pm 7} = \pm 1/7\pi, \ldots$

It follows from Parseval's formula that

$$\begin{aligned} P_{av} &= \sum_{m=-\infty}^{\infty} c_m c_m^* \\ &= 0.5^2 + \sum_{\substack{-\infty \\ m \text{ odd}}}^{\infty} \left(\frac{(-1)^{(m-1)/2}}{m\pi} \right)^2 \\ &= 0.25 + 2 \sum_{\substack{m=1 \\ m \text{ odd}}}^{\infty} \frac{1}{(m\pi)^2} \\ &= 0.25 + \frac{2}{\pi^2} + \frac{2}{9\pi^2} + \frac{2}{25\pi^2} + \ldots \\ &= 0.5 \end{aligned}$$

If can be seen that the total average power in the signal is 0.5 and the zero-frequency component of the signal contributes a power of 0.25. The power contributed by the fundamental frequency component is $2 \times 1/\pi^2 \approx 0.202$. All the remaining frequency components contribute less than 10% (0.05) of the total power.

The advantage of the Fourier series description is that it associates signal power with frequency. When transmitting data, we need to make sure that most of the signal energy is transmitted.

In the case of the train of pulses, it is sufficient to transmit just the fundamental frequency. The higher harmonics contribute little to the energy of the signal.

Also, since the zero frequency component conveys no useful information (it is just a constant), then this analysis shows that it should be possible to halve the energy requirement of the system by not transmitting it.