23 Fourier Transform

Up to this point, only periodic signals have been considered. Aperiodic signals can also be the subject of Fourier analysis.

Recall that the Fourier Series representation of a periodic signal $f(t)$ with period $P = 2\pi/\omega_o$ is

$$\begin{aligned} f(t) &= \sum_{m=-\infty}^{\infty} c_m e^{jm\omega_o t} \\ c_m &= \frac{1}{P} \int_{<P>} f(t) e^{-jm\omega_o t}\, dt \end{aligned}$$

This can be extended to the aperiodic case by making P very large. In the infinite case, the Fourier coefficients, c_m, form a continuum, which is called a *Frequency Spectrum*.

To develop this define $F(m\omega_o) = c_m P$, and $\omega = m\omega_o$, then

$$F(\omega) = \int_{-P/2}^{P/2} f(t)e^{-j\omega t}\,dt$$

so the Fourier series becomes

$$f(t) = \sum_{m=-\infty}^{\infty} \frac{F(\omega)}{P} e^{j\omega t}$$

Noting $m = \omega/\omega_o$, defining $\omega_o = d\omega$ and making it small, makes $P = 2\pi/\omega_o$ very large. The Fourier series becomes a continuum and the summation an integration:

$$\begin{aligned} f(t) &= \sum_{\omega/\omega_o=-\infty}^{\infty} \frac{F(\omega)\omega_o}{2\pi} e^{j\omega t} \\ &= \sum_{\omega/d\omega=-\infty}^{\infty} \frac{F(\omega)\,d\omega}{2\pi} e^{j\omega t} \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{j\omega t}\,d\omega \end{aligned}$$

The result is the

Fourier Transform Pair

$$\begin{aligned} F(\omega) &= \int_{-\infty}^{\infty} f(t)e^{-j\omega t}\,dt \\ f(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{j\omega t}\,d\omega \end{aligned}$$

$F(\omega)$ is called the *Fourier transform* of $f(t)$.

The process is defined as an operator on functions, so that

$$F(\omega) = \mathcal{F}\left[f(t)\right] \qquad \text{and} \qquad f(t) = \mathcal{F}^{-1}\left[F(\omega)\right].$$

23.1 Some Fourier Transform Pairs

Time-Domain Impulse

Consider the time-domain impulse function $f(t) = \delta(t - t_o)$.

Its Fourier transform is

$$\begin{aligned} F(\omega) &= \int_{-\infty}^{\infty} \delta(t - t_o) e^{-j\omega t}\, dt \\ &= e^{-j\omega t_o} \end{aligned}$$

Thus the Fourier transform of an impulse has unit magnitude for all ω. If $t_o = 0$ then $\mathcal{F}[\delta(t)] = 1$.

The significant observation is that when the signal is confined to a narrow pulse in the time-domain it is spread evenly over all frequencies in frequency-domain.

Frequency-Domain Impulse

Consider the impulse function in the frequency domain $F(\omega) = \delta(\omega - \omega_o)$. Its inverse Fourier transform is

$$\begin{aligned} f(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \delta(\omega - \omega_o)e^{j\omega t}\,d\omega \\ &= \frac{1}{2\pi}e^{j\omega_o t} \end{aligned}$$

Thus an impulse in the frequency domain is a complex exponential in the time domain, which has unity magnitude over all time. If $\omega_o = 0$ then $f(t) = 1/2\pi$.

The significant observation is that when the signal is confined to a narrow pulse in the frequency-domain it is spread evenly in the time-domain over all time.

Two Frequency-domain Impulses

The case of $F(\omega) = \delta(\omega + \omega_o) + \delta(\omega - \omega_o)$ gives an inverse Fourier transform of

$$\begin{aligned} f(t) &= \mathcal{F}^{-1}\left[\delta(\omega + \omega_o) + \delta(\omega - \omega_o)\right] \\ &= \frac{1}{2\pi}e^{-j\omega_o t} + \frac{1}{2\pi}e^{j\omega_o t} \\ &= \frac{1}{\pi}\cos\omega_o t \end{aligned}$$

This confirms that a cosine (or sinusoidal) signal is the sum of complex exponentials with positive and negative frequencies.

23.2 Some Properties of the Fourier Transform

The Fourier transform has some of the same properties as the discrete Fourier series of periodic signals. For example, the Fourier transform of a real function has complex symmetry.

That is, given $F(\omega) = \mathcal{F}[f(t)]$, where $f(t) = f^*(t)$ (that is, f is real), then

The magnitude of the frequency spectrum is even, that is

$$|F(\omega)| = |F(-\omega)|$$

The phase of the frequency spectrum is odd, that is

$$\angle F(\omega) = -\angle F(-\omega)$$

23.2.1 Time Reversal

Consider $F(\omega) = \mathcal{F}[f(t)]$. If time t is reversed, then the Fourier transform of $f(-t)$ is

$$\begin{aligned}
\mathcal{F}[f(-t)] &= \int_{-\infty}^{\infty} f(-t)e^{-j\omega t}\,dt \\
&= -\int_{\infty}^{-\infty} f(u)e^{j\omega u}\,du \qquad \text{letting } u = -t \\
&= \int_{-\infty}^{\infty} f(u)e^{-j(-\omega)u}\,du \\
&= F(-\omega)
\end{aligned}$$

This gives the time reversal property: a reversal in the time domain is a reversal in the frequency domain.

In other words, if signals with positive frequency travel forward in time, then signals with negative frequencies travel backward in time.

23.2.2 Time Shift

A shift and scale of the time variable results in a scale and phase change in the Fourier transform. To see this, consider $F(\omega) = \mathcal{F}[f(t)]$, then the Fourier transform of $f(t - t_o)$ becomes

$$
\begin{aligned}
\mathcal{F}[f(t - t_o)] &= \int_{-\infty}^{\infty} f(t - t_o) e^{-j\omega t}\, dt \\
&= \int_{-\infty}^{\infty} f(u) e^{-j\omega(u + t_o)}\, du \qquad \text{letting } u = t - t_o \\
&= e^{-j\omega t_o} \int_{-\infty}^{\infty} f(u) e^{-j\omega u}\, du \\
&= e^{-j\omega t_o} \mathcal{F}[f(t)]
\end{aligned}
$$

This gives the time shift property: a shift in the time domain is a *phase winding* in the frequency domain:

$$
\boxed{\mathcal{F}[f(t - t_o)] = e^{-j\omega t_o} \mathcal{F}[f(t)]}
$$

23.2.3 Time Scale

Consider $F(\omega) = \mathcal{F}[f(t)]$. If time t is stretched by factor a, then the Fourier transform of $f(at)$ is

$$
\begin{aligned}
\mathcal{F}[f(at)] &= \int_{-\infty}^{\infty} f(at)e^{-j\omega t}\,dt \\
&= \int_{-\infty}^{\infty} f(u)e^{-j\omega u/a}\,du/a \qquad \text{letting } u = at \\
&= \frac{1}{a}\int_{-a\infty}^{a\infty} f(u)e^{-j(\omega/a)u}\,du \\
&= \frac{1}{|a|}\int_{-\infty}^{\infty} f(u)e^{-j(\omega/a)u}\,du \\
&= \frac{1}{|a|}F\left(\frac{\omega}{a}\right)
\end{aligned}
$$

This gives the time stretch property: a stretch in the time domain is a *frequency compression* in the frequency domain:

If $F(\omega) = \mathcal{F}[f(t)]$ then

$$\mathcal{F}[f(at)] = \frac{1}{|a|}F\left(\frac{\omega}{a}\right)$$

It is interesting to note that a signal stretched out over a long time (a small) becomes confined to a smaller range of frequencies.

23.2.4 Duality

There is an obvious symmetry in the Fourier transform pair, which is formally noted as a duality between the time and frequency domains.

Given that

$$F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}\,dt \qquad f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{j\omega t}\,d\omega$$

That is:

$$F(\omega) = \mathcal{F}[f(t)] \qquad \text{and} \qquad f(t) = \mathcal{F}^{-1}[F(\omega)].$$

The dual statement is

$$f(\omega) = \mathcal{F}[F(-t)]/2\pi \qquad \text{and} \qquad F(t) = 2\pi\mathcal{F}^{-1}[f(-\omega)]$$

To derive this, first swap the variables ω and t in the pair of equations. Then rearrange the pair of equations to resemble the inverse operations:

$$
\begin{aligned}
f(\omega) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} F(t)e^{jt\omega}\,dt \\
&= \frac{1}{2\pi}\left[\int_{-\infty}^{\infty} F(t)e^{j\omega t}\,dt\right] \\
&= \frac{1}{2\pi}\left[-\int_{\infty}^{-\infty} F(-t)e^{-j\omega t}\,dt\right] \\
&= \frac{1}{2\pi}\left[\int_{-\infty}^{\infty} F(-t)e^{-j\omega t}\,dt\right] \\
&= \frac{1}{2\pi}\mathcal{F}\left[F(-t)\right]
\end{aligned}
$$

$$
\begin{aligned}
F(t) &= \int_{-\infty}^{\infty} f(\omega)e^{-jt\omega}\,d\omega \\
&= 2\pi\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} f(\omega)e^{-j\omega t}\,d\omega\right] \\
&= 2\pi\left[-\frac{1}{2\pi}\int_{\infty}^{-\infty} f(-\omega)e^{j\omega t}\,d\omega\right] \\
&= 2\pi\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} f(-\omega)e^{j\omega t}\,d\omega\right] \\
&= 2\pi\mathcal{F}^{-1}\left[f(-\omega)\right]
\end{aligned}
$$

23.2.5 Frequency Shift

Frequency shift property:

If $F(\omega) = \mathcal{F}[f(t)]$ then

$$\mathcal{F}\left[f(t)e^{j\omega_o t}\right] = F(\omega - \omega_o)$$

This is easily shown

$$\begin{aligned}
\mathcal{F}\left[f(t)e^{j\omega_o t}\right] &= \int_{-\infty}^{\infty} f(t)e^{j\omega_o t}e^{-j\omega t}\,dt \\
&= \int_{-\infty}^{\infty} f(t)e^{-j(\omega-\omega_o)t}\,dt \\
&= F(\omega - \omega_o)
\end{aligned}$$

23.3 Summary

A few of the properties of the Fourier transform, $F(\omega) = \mathcal{F}[f(t)]$, that are useful in signal analysis are:

Linearity

Time function	$\alpha_1 f_1(t) + \alpha_2 f_2(t)$
Fourier transform	$\alpha_1 F_1(\omega) + \alpha_2 F_2(\omega)$
Implication	It is possible to decompose analysis into simpler parts.

Frequency Shifting

Time function	$f(t)\, e^{j\omega t}$
Fourier transform	$F(\omega - \omega_o)$
Implication	It is possible to work at base band ($\omega_o = 0$).

This is exactly the operation of amplitude modulation where an audio signal is shifted to become a radio frequency.

Time Shifting

Time function	$f(t - t_o)$
Fourier transform	$\epsilon^{-j\omega t}\, F(\omega)$
Implication	Magnitude of the frequency domain is independent of time shifting.

Time Scaling

Time function	$f(at)$
Fourier transform	$\frac{1}{\lvert a \rvert} F(\frac{\omega}{a})$
Implication	Fast time signals require wide bandwidth.

Time Reversal

Time function	$f(-t)$
Fourier transform	$F(-\omega)$
Implication	Negative frequency signals are reversed in time.

Duality

Time function	$F(\omega) = \mathcal{F}[f(t)]$
Fourier transform	$f(-\omega) = \frac{1}{2\pi}\mathcal{F}[F(t)]$
Implication	Time-frequency relationships exist in reverse.

23.4 Rectangular Pulse

As and example, consider a one time rectangular pulse defined by

$$f(t) = \begin{cases} 1 & -1 \leq t \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Its Fourier transform is given by

$$\begin{aligned} F(\omega) &= \int_{-\infty}^{\infty} f(t) e^{-j\omega t}\, dt \\ &= \int_{-1}^{1} e^{-j\omega t}\, dt \\ &= \frac{1}{-j\omega}\left[e^{-j\omega} - e^{j\omega}\right] \\ &= \frac{2}{\omega}\sin\omega \end{aligned}$$

The rectangular pulse in the time domain has the form $(\sin x)/x$ in the frequency domain.

This gives the Fourier transform pair

$$f(t) = \begin{cases} 1 & -1 \leq t \leq 1 \\ 0 & \text{otherwise} \end{cases} \qquad F(\omega) = \mathcal{F}\left[f(t)\right] = \frac{2\sin\omega}{\omega}$$

The duality property

$$f(-\omega) = \frac{1}{2\pi}\mathcal{F}\left[F(t)\right]$$

then gives other Fourier transform pairs

$$F(t) = \frac{2\sin t}{t} \qquad f(-\omega) = \tfrac{1}{2\pi}\begin{cases} 1 & -1 \leq \omega \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

or

$$F(t) = \frac{2\sin t}{t} \qquad f(\omega) = \tfrac{1}{2\pi}\begin{cases} 1 & -1 \leq \omega \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

or

$$F(t) = \frac{\sin t}{\pi t} \qquad f(\omega) = \begin{cases} 1 & -1 \leq \omega \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

The rectangular pulse in the frequency domain has the form $(\sin x)/x$ in the time domain.