24 Aperiodic Convolution

Consider the time-domain convolution of $h(t)$ and $u(t)$

$$\begin{aligned}
y(t) &= h(t) * u(t) \\
&= \int\limits_{-\infty}^{\infty} h(t-\tau)u(\tau)\, d\tau \\
&= \int\limits_{-\infty}^{\infty} h(\tau)u(t-\tau)\, d\tau
\end{aligned}$$

The Fourier transform of $y(t)$ is

$$\begin{aligned}
Y(\omega) &= \mathcal{F}\left[y(t)\right] \\
&= \int\limits_{-\infty}^{\infty} y(t)e^{-j\omega t}\, dt \\
&= \int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} h(t-\tau)u(\tau)\, d\tau\; e^{-j\omega t}\, dt \\
&= \int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} h(t-\tau)u(\tau)\, d\tau\; e^{-j\omega(t-\tau)}e^{-j\omega\tau}\, dt \\
&= \int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} h(t-\tau)e^{-j\omega(t-\tau)}\, dt\; u(\tau)e^{-j\omega\tau}\, d\tau \\
&= \int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} h(t')e^{-j\omega t'}\, dt'\; u(\tau)e^{-j\omega\tau}\, d\tau \qquad\qquad t' = t-\tau \\
&= \int\limits_{-\infty}^{\infty} \mathcal{F}\left[h(t')\right] u(\tau)e^{-j\omega\tau}\, d\tau \\
&= \mathcal{F}\left[h(t')\right]\int\limits_{-\infty}^{\infty} u(\tau)e^{-j\omega\tau}\, d\tau \\
&= \mathcal{F}\left[h(t')\right]\mathcal{F}\left[u(t)\right] \\
&= H(\omega)U(\omega)
\end{aligned}$$

Convolution in the time domain is multiplication in the frequency domain.

The property of duality implies the inverse as well. That is, multiplication in the time domain is convolution in the frequency domain.

Consider the frequency-domain convolution of $H(\omega)$ and $U(\omega)$

$$\begin{aligned} Y(\omega) &= H(\omega) * U(\omega) \\ &= \int_{-\infty}^{\infty} H(\omega-\lambda)U(\lambda)\,d\lambda \end{aligned}$$

The inverse Fourier transform of $Y(\omega)$ is

$$\begin{aligned} y(t) &= \mathcal{F}^{-1}\left[Y(\omega)\right] \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} Y(\omega)e^{j\omega t}\,d\omega \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} H(\omega-\lambda)U(\lambda)\,d\lambda\ e^{j\omega t}\,d\omega \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} H(\omega-\lambda)U(\lambda)\,d\lambda\ e^{j(\omega-\lambda)t}e^{j\lambda t}\,d\omega \\ &= \int_{-\infty}^{\infty}\frac{1}{2\pi}\int_{-\infty}^{\infty} H(\omega-\lambda)\,e^{j(\omega-\lambda)t}d\omega\ U(\lambda)e^{j\lambda t}\,d\lambda \\ &= \int_{-\infty}^{\infty}\frac{1}{2\pi}\int_{-\infty}^{\infty} H(\omega')\,e^{j\omega' t}d\omega'\ U(\lambda)e^{j\lambda t}\,d\lambda \qquad \omega' = \omega-\lambda \\ &= \int_{-\infty}^{\infty} \mathcal{F}^{-1}\left[H(\omega')\right]\ U(\lambda)e^{j\lambda t}\,d\lambda \\ &= 2\pi\mathcal{F}^{-1}\left[H(\omega')\right]\ \frac{1}{2\pi}\int_{-\infty}^{\infty} U(\lambda)e^{j\lambda t}\,d\lambda \\ &= 2\pi\mathcal{F}^{-1}\left[H(\omega')\right]\mathcal{F}^{-1}\left[U(\lambda)\right] \\ &= 2\pi h(t)u(t) \end{aligned}$$

Convolution in the frequency domain is multiplication in the time domain.

24.1 Low-pass Filter

An ideal low-pass filter is a system whose output is the components of the input that have a frequency less than the cut-off frequency of the filter ω_c. That is, if the spectrum of the input is given by $U(\omega) = \mathcal{F}[u(t)]$ then the spectrum of the output produced by the filter is

$$Y(\omega) = H(\omega)U(\omega)$$

where

$$H(\omega) = \begin{cases} 1 & |\omega| \le \omega_c \\ 0 & |\omega| > \omega_c \end{cases}$$

The Fourier transform pair established in the rectangular pulse example of section 23.4,

$$f(t) = \frac{\sin t}{\pi t} \qquad F(\omega) = \begin{cases} 1 & -1 \le \omega \le 1 \\ 0 & \text{otherwise} \end{cases}$$

gives, after applying the time scaling property, the transform pair

$$h(t) = \frac{\sin \omega_c t}{\pi t} \qquad H(\omega) = \begin{cases} 1 & |\omega| \le \omega_c \\ 0 & |\omega| > \omega_c \end{cases}$$

Now the frequency-domain relationship

$$Y(\omega) = H(\omega)U(\omega)$$

is a convolution in the time domain

$$y(t) = h(t) * u(t)$$

This implies that the impulse response ($u(t) = \delta(t)$) of the ideal low-pass filter is

$$h(t) = \frac{\sin \omega_c t}{\pi t}$$

Note that this is a non-causal system and can therefore never be realised in practice.

24.2 Periodic Signals

24.2.1 Fourier Transform of periodic signals

Consider a periodic signal $f(t)$ that has period $P = 2\pi/\omega_o$, with Fourier series representation

$$f(t) = \sum_{m=-\infty}^{\infty} c_m e^{jm\omega_o t} \qquad\qquad c_m = \frac{1}{P}\int_{<P>} f(t)e^{-jm\omega_o t}\,dt$$

The Fourier transform of this periodic signal is

$$\begin{aligned}
F(\omega) &= \mathcal{F}[f(t)] \\
&= \int_{-\infty}^{\infty} f(t)e^{-j\omega t}\,dt \\
&= \int_{-\infty}^{\infty}\left[\sum_{m=-\infty}^{\infty} c_m e^{jmw_o t}\right] e^{-jwt}\,dt \\
&= \sum_{m=-\infty}^{\infty} c_m \int_{-\infty}^{\infty} e^{jmw_o t}e^{-jwt}\,dt \\
&= \sum_{m=-\infty}^{\infty} c_m \mathcal{F}\left[e^{jm\omega_o t}\right] \\
&= 2\pi \sum_{m=-\infty}^{\infty} c_m \delta(\omega - m\omega_o)
\end{aligned}$$

since $\mathcal{F}[e^{jm\omega_o t}] = 2\pi\delta(\omega - m\omega_o)$ (see section 23.1).

Thus the Fourier transform of a periodic signal is a sequence of impulses with values proportional to the Fourier coefficients of the signal's Fourier series.

The Fourier transform of a periodic signal $f(t)$ with period $P = 2\pi/\omega_o$ is

$$F(\omega) = 2\pi \sum_{m=-\infty}^{\infty} c_m \delta(\omega - m\omega_o)$$

where

$$c_m = \frac{1}{P}\int_{<P>} f(t)e^{-jm\omega_o t}\,dt$$

Example

Consider a periodic sequence of impulses, with $P = 2\pi/\omega_o$

$$f(t) = \sum_{n=-\infty}^{\infty} \delta(t - nP)$$

The Fourier transform is given by

$$F(\omega) = 2\pi \sum_{m=-\infty}^{\infty} c_m \delta(\omega - m\omega_o)$$

where

$$\begin{aligned}
c_m &= \frac{1}{P} \int_{<P>} f(t) e^{-jm\omega_o t} \, dt \\
&= \frac{1}{P} \int_{nP-P/2}^{nP+P/2} \sum_{n=-\infty}^{\infty} \delta(t - nP) \, e^{-jm\omega_o t} \, dt \\
&= \frac{1}{P} \int_{-\infty}^{\infty} \delta(t - nP) \, e^{-jm\omega_o t} \, dt \\
&= \frac{1}{P} e^{-jm\omega_o nP} \\
&= \frac{1}{P} e^{-j2\pi mn} \\
&= \frac{1}{P}
\end{aligned}$$

Thus the Fourier transform pair for a periodic sequence of impulses is

$$f(t) = \sum_{n=-\infty}^{\infty} \delta(t - nP) \qquad\qquad F(\omega) = \frac{2\pi}{P} \sum_{m=-\infty}^{\infty} \delta\left(\omega - m\frac{2\pi}{P}\right)$$

24.3 Sampling

In many applications, in particular computer or digital processing, it is necessary to sample a continuous-time signal $f(t)$ to produce a discrete-time signal $f_s(t)$. In most cases, the discrete-time signal is a sequence of values given by the sampling $f(kT)$ where k is an integer and T is the period of the sampling.

Sampling is defined as multiplication in the time domain by a train of impulses.

$$f_s(t) = f(t) \sum_{k=-\infty}^{\infty} \delta(t - kT),$$

which can be viewed in the frequency domain as a convolution.

$$\mathcal{F}\left[f_s(t)\right] = \frac{1}{2\pi}\left\{\mathcal{F}[f(t)] * \mathcal{F}\left[\sum_{k=-\infty}^{\infty} \delta(t-kT)\right]\right\}.$$

Given that $\mathcal{F}[f(t)] = F(\omega)$ and $\mathcal{F}\left[\sum_{k=-\infty}^{\infty} \delta(t-kT)\right] = \frac{2\pi}{T}\sum_{m=-\infty}^{\infty}\delta(\omega - m\frac{2\pi}{T})$ then the Fourier transform of the sampled signal is

$$\begin{aligned}
F_s(\omega) &= \mathcal{F}[f_s(t)] \\
&= \frac{1}{2\pi}\mathcal{F}[f(t)] * \mathcal{F}\left[\sum_{k=-\infty}^{\infty} \delta(t-kT)\right] \\
&= \frac{1}{2\pi}F(\omega) * \frac{2\pi}{T}\sum_{m=-\infty}^{\infty}\delta\left(\omega - m\frac{2\pi}{T}\right) \qquad \swarrow \omega_s \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\lambda)\,\frac{2\pi}{T}\sum_{m=-\infty}^{\infty}\delta\left(\omega - \lambda - m\frac{2\pi}{T}\right) d\lambda \\
&= \frac{1}{T}\sum_{m=-\infty}^{\infty}\int_{-\infty}^{\infty} F(\lambda)\,\delta\left(\omega - \lambda - m\frac{2\pi}{T}\right) d\lambda \\
&= \frac{1}{T}\sum_{m=-\infty}^{\infty} F\left(\omega - m\frac{2\pi}{T}\right)
\end{aligned}$$

Thus

Given the Fourier transform of a continuous-time signal $\mathcal{F}[f(t)] = F(\omega)$ then the Fourier transform pair for the sampling of the signal is

$$f_s(t) = f(t) \sum_{k=-\infty}^{\infty} \delta(t - kT)$$

$$F_s(\omega) = \frac{1}{T} \sum_{m=-\infty}^{\infty} F\left(\omega - m\frac{2\pi}{T}\right)$$

Note that $F_s(\omega)$ is periodic with period $\omega_s = 2\pi/T$.

Proof

$$\begin{aligned} F_s(\omega) &= \frac{1}{T} \sum_{m=-\infty}^{\infty} F(\omega - m\omega_s) \\ &= \frac{1}{T} \sum_{m=-\infty+1}^{\infty+1} F(\omega + \omega_s - m\omega_s) \\ &= F_s(\omega + \omega_s) \end{aligned}$$

◇

This implies that the spectrum is completely known if it is defined over one period, say over the period $0 \leq \omega < \omega_s$, or alternatively over the period $-\omega_s/2 < \omega \leq \omega_s/2$.

This is an important point because if the unsampled signal is to be recovered from the sampled signal then $F(\omega)$ must be zero outside the interval $-\omega_s/2 < \omega \leq \omega_s/2$. That is, zero when $-\pi < \omega T \leq \pi$, (c.f. the definition of the frequency of discrete-time signals).

This leads again to the sampling theorem presented in section 16.6.2.

24.3.1 Sampling Theorem

Let $f(t)$ be a continuous-time signal whose frequency spectrum is band limited to W (rad/s). Then $f(t)$ can be recovered from its sampled sequence $f(kT)$ if the sampling period T is smaller than π/W. That is if $\omega_s = 2\pi/T$ is greater than $2W$.

Example

Consider a sinusoidal signal and its Fourier transform

$$\begin{aligned} f(t) &= 2\cos 2\pi t \\ F(\omega) &= \delta(\omega - 2\pi) + \delta(\omega + 2\pi) \end{aligned}$$

This signal is band-limited to 2π rad/s.

Sampling this signal in the time domain at a rate $\omega_s = 2N\pi$ gives

$$f_s(t) = 2\cos 2\pi t \sum_{k=-\infty}^{\infty} \delta(t - k/N)$$

N is the number of samples per period of $f(t)$. The sampling period is $T = 1/N$, which should be less than $1/2$ a second (i.e. $N > 2$).

The Fourier transform of the sampled signal is

$$\begin{aligned} F_s(\omega) &= N \sum_{m=-\infty}^{\infty} F(\omega - m2\pi/T) \\ &= N \sum_{m=-\infty}^{\infty} \delta(\omega - 2\pi mN - 2\pi) + \delta(\omega - 2\pi nM + 2\pi) \\ &= N \sum_{m=-\infty}^{\infty} \delta(\omega - 2\pi(mN + 1)) + \delta(\omega - 2\pi(nM - 1)) \end{aligned}$$

This is a sum of the original Fourier transform $F(\omega)$ — the term corresponding to $m = 0$ — and copies of it shifted in the frequency domain by $m\omega_s$.

If $N > 2$ then the only terms of the summation that appear in the interval $-N\pi < \omega \leq N\pi$ (i.e. for $-\pi < \omega T \leq \pi$) are those for $m = 0$. That is, only the original Fourier transform $F(\omega)$ occupies the interval $-N\pi < \omega \leq N\pi$.

For example, if $N = 3$ then the Fourier transform consists of impulses at frequencies

$$\ldots, -14\pi, -10\pi, -8\pi, -4\pi, -2\pi, 2\pi, 4\pi, 8\pi, 10\pi, 14\pi, \ldots$$

of which only those at -2π and 2π appear in the interval $-3\pi < \omega \leq 3\pi$. If all frequencies outside this interval are removed by a low-pass filter, then the original signal $f(t)$ is recovered.

If $N < 2$ then terms in the summation for $m \neq 0$ will appear in the interval $-N\pi < \omega \leq N\pi$. That is, a shifted copy of the original spectrum will occupy some part of this interval.

For example, if $N = 3/2$ then the Fourier transform consists of impulses at frequencies

$$\ldots, -7\pi, -5\pi, -4\pi, -2\pi, -\pi, \pi, 2\pi, 4\pi, 5\pi, 7\pi, \ldots$$

of which only those at $-\pi$ and π, appear in the interval $-3N\pi/2 < \omega \leq 3N\pi/2$. These correspond to $m = -1$ and 1, respectively.

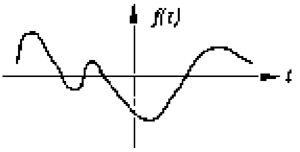
If all frequencies outside the interval $-3N\pi/2 < \omega \leq 3N\pi/2$ are removed by a low-pass filter, then the signal recovered will be $2 cos \pi t$, which is at a different frequency to that of the original signal.

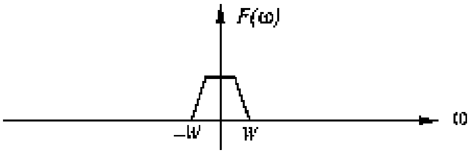
This change in frequency is called *aliasing*. The original cosine signal at a frequency 2π is aliased to a frequency of π when an attempt is made to recover it from the sampled data.

The signal that is actually recovered is a copy of the original, which has been shifted in the frequency domain.

24.3.2 Aliasing

Consider a continuous-time signal $f(t)$ with a spectrum $F(\omega)$, which is band limited to $|\omega| \leq W$.

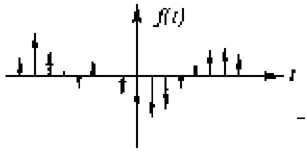


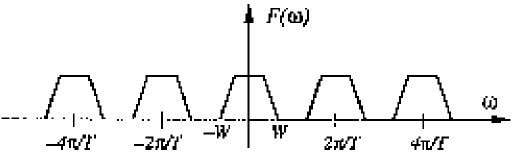


When sampled at a rate $2\pi/T$, which is greater than $2W$ a signal is created with a spectrum that is the sum of copies of $F(\omega)$ centered on frequencies of $2\pi m/T$:

$$f_s(t) = f(t) \sum_{k=-\infty}^{\infty} \delta(t - kT)$$

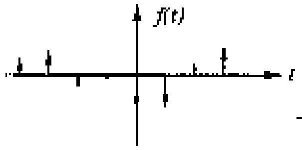
$$F_s(\omega) = \frac{1}{T} \sum_{m=-\infty}^{\infty} F\left(\omega - m\frac{2\pi}{T}\right)$$

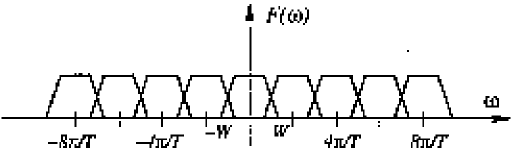




The original spectrum is contained within the interval $-\pi/T < \omega \leq \pi/T$, so the original signal can be recovered.

If the same rate is reduced, so that $\pi/T < W$ then the copies of $F(\omega)$ will overlap





The original spectrum no longer exists in any interval, so the original signal cannot be recovered.