25 Discrete Fourier Transform

Analysis of continuous-time signals with a computer requires two alterations. First, the signal must be sampled and second the number of samples is restricted (by the amount of memory if nothing else). Thus any computer based Fourier analysis on continuous-time signals is preceded by construction of a *finite length sample sequence*. Usually, but not necessarily always, the sampling is performed at regular intervals.

25.1 Discrete Data

Recall two important results from the previous section.

- The Fourier transform of a periodic time-domain signal with period $P = 2\pi/\omega_o$ is a sum of impulses, which are separated in frequency by $\omega_o = 2\pi/P$.
- The Fourier transform of a sampled time-domain signal that is a sum of impulses separated in time by $T = 2\pi/\omega_s$ is a periodic signal in the frequency domain with period $\omega_s = 2\pi/T$.

Either of these results implies the other by the duality property of the Fourier transform.

> If both the time and frequency domains are discrete then both domains must be periodic.

Discrete data in both the time and frequency domains is required for computer processing. The time-domain sequence is a list of the values of the impulses that result from sampling a signal, and the second result implies that the frequency-domain view of this sampled signal must be periodic. The frequency-domain sequence is also a list of sizes of impulses, so the first result implies that the time-domain view must be periodic.

25.1.1 Time and Frequency Domains

Consider a sample sequence constructed by taking N samples of a signal $f(t)$ at a regular sampling interval of T. The frequency-domain is periodic with period $\omega_s = 2\pi/T$ because it is copies of the spectrum of $f(t)$ shifted by multiples of ω_s.

The longest period in the time-domain samples is $P = NT$. This corresponds to the lowest observable frequency, so the frequency domain impulses cannot be separated by less than $\omega_0 = 2\pi/P$.

Thus there are at most $\omega_s/\omega_0 = N$ values in one period of the frequency-domain.

The implication is that a sequence of N values derived from sampling a signal at a regular interval T has a related frequency-domain sequence of N values, which correspond to a sampling in the frequency domain at a regular interval $2\pi/P$ (where $P = NT$).

25.2 Discrete Fourier Transform

Consider a discrete signal $f[k] = f(kT)$, which has an envelope

$$f_s(t) = \sum_{k=-\infty}^{\infty} f(t)\,\delta(t - kT).$$

This is a sampling of $f(t)$ at rate $\omega_s = 2\pi/T$.

The Fourier transform $F_s(\omega)$ of $f_s(t)$ is periodic with period ω_s. It is reasonable to expect that only one period of $F_s(\omega)$ is sufficient to describe the sampled signal.

To develop this, describe $f(t)$ in terms of its Fourier transform:

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{j\omega t}\, d\omega$$

Dividing the integration into a sum of integrations each over interval ω_s gives

$$\begin{aligned} f(t) &= \frac{1}{2\pi} \sum_{m=-\infty}^{\infty} \int_0^{\omega_s} F(\omega - m\omega_s) e^{j(\omega - m\omega_s)t}\, d\omega \\ &= \frac{1}{2\pi} \int_0^{\omega_s} \left(\sum_{m=-\infty}^{\infty} F(\omega - m\omega_s) e^{-jm\omega_s t} \right) e^{j\omega t}\, d\omega \end{aligned}$$

Setting $t = kT$ and $\omega_s = 2\pi/T$ gives

$$\begin{aligned} f(kT) &= \frac{1}{2\pi} \int_0^{\omega_s} \left(\sum_{m=-\infty}^{\infty} F(\omega - m\omega_s) e^{-jm(2\pi/T)kT} \right) e^{j\omega kT}\, d\omega \\ &= \frac{T}{2\pi} \int_0^{\omega_s} \left(\frac{1}{T} \sum_{m=-\infty}^{\infty} F(\omega - m\omega_s) \right) e^{j\omega kT}\, d\omega \\ &= \frac{T}{2\pi} \int_0^{\omega_s} F_s(\omega) e^{j\omega kT}\, d\omega \end{aligned}$$

The integral can be approximated by a summation over intervals of $d\omega = \omega_o$, so that discrete values of $F_S(\omega)$ at $\omega = m\omega_o$ where $\omega_o = 2\pi/NT$, are used.

$\omega = m\omega_o$
$d\omega = \omega_o = \frac{2\pi}{NT}$

$$
\begin{aligned}
f(kT) &= \frac{T}{2\pi}\int_0^{\omega_s} F_s(\omega)e^{jm\omega_o kT}\,d\omega \\
&\approx \frac{T}{2\pi}\sum_{m=0}^{N-1} F_s(m\omega_o)e^{jmkT2\pi/NT}\,\frac{2\pi}{NT} \\
&= \frac{1}{N}\sum_{m=0}^{N-1} F_s(m\omega_o)e^{j2\pi mk/N}
\end{aligned}
$$

Now this is the basis for defining sequences $f[k] = f(kT)$ and $F[m] = F_s(m\omega_o)$ and the relationship:

$$
f[k] = \frac{1}{N}\sum_{m=0}^{N-1} F[m]e^{j2\pi mk/N}
$$

This can now be inverted to determine $F[m]$ by taking another summation

$$
\begin{aligned}
\sum_{k=0}^{N-1} f[k]e^{-j2\pi mk/N} &= \sum_{k=0}^{N-1}\left(\frac{1}{N}\sum_{n=0}^{N-1} F[n]e^{j2\pi nk/N}\right)e^{-j2\pi mk/N} \\
&= \frac{1}{N}\sum_{k=0}^{N-1}\sum_{n=0}^{N-1} F[n]e^{j2\pi(n-m)k/N} \\
&= \frac{1}{N}\sum_{n=0}^{N-1} F[n]\sum_{k=0}^{N-1} e^{j2\pi(n-m)k/N} \\
&= \frac{1}{N}\sum_{n=0}^{N-1} F[n]\,N\delta(n-m) \\
&= F[m]
\end{aligned}
$$

This establishes the *Discrete Fourier Transform* (DFT) pair:

$$
\boxed{
\begin{aligned}
f[k] &= \frac{1}{N}\sum_{m=0}^{N-1} F[m]e^{j2\pi mk/N} & 0 \le k < N \\
F[m] &= \sum_{k=0}^{N-1} f[k]e^{-j2\pi mk/N} & 0 \le m < N
\end{aligned}
}
$$

25.3 DFT versus FT

Although the DFT uses only N terms, the sequence $f[k]$ is periodic with period N, so that $f[k] = f[k+N]$. This implies that if $f[k] = f(kT)$ then $f(t)$ can be periodic with period NT, so it can be represented by a Fourier series with fundamental frequency $\omega_o = 2\pi/NT$. The coefficients of this series are $c_m = F[m]/N$.

The sequence $F[m]$ is also periodic with period N, so that $F[m] = F[m+N]$. This means that $F[-m] = F[N-m]$. Although the DFT is defined for positive m, both positive and negative frequencies are represented.

25.3.1 Sampling Rate and Length

Continuous-time Fourier transforms can be approximated by sampling the continuous-time signals and then using the Discrete Fourier Transform. The time-domain sequence is simply $f[k] = f(kT)$. The frequency-domain sequence is $F[m] = F_s(m\omega_o)$, which if no aliasing occurs, gives

$$F(m\omega_o) = TF[m]$$

and

$$F(-m\omega_o) = TF[N-m]$$

for $0 \leq m \leq N/2$.

The maximum frequency magnitude correctly represented is $N\omega_o/2 = \omega_s/2$, which is consistent with the sampling theorem.

The important result is that the sampling rate $1/T$ should be more than twice the highest frequency in the signal's spectrum.

The number of samples N sets the frequency resolution to $\omega_o = 2\pi/NT$, which should be a multiple of the fundamental frequency of the time-domain signal.

Example

Consider the continuous-time signal

$$f(t) = 2\cos 300\pi t + 2\sin 400\pi t.$$

Its Fourier transform is

$$F(\omega) = j\delta(\omega + 400\pi) + \delta(\omega + 300\pi) + \delta(\omega - 300\pi) - j\delta(\omega - 400\pi)$$

The signal's fundamental frequency is 100π rad/s to which $\omega_o = 2\pi/NT$ can be set.

The highest frequency in the signal is 400π, so $2\pi/T$ must be at least 800π.

Choosing $T = 2$ ms, which is less than $2\pi/800\pi$, gives $N = 10$.

This gives

$$f[k] = \{2.0,\ 0.558,\ -3.52,\ 3.52,\ -0.558,\ 2.0,\ 1.794,\ -0.284,\ 0.284,\ 1.794\}$$

$$F[m] = \{0.0,\ 0.0,\ 0.0,\ 500,\ -500j,\ 0.0,\ 500j,\ 500,\ 0.0,\ 0.0\}$$

Note that $F[m] = F(m\omega_o)/T$, and the frequencies corresponding to $m = 0, 1, \ldots, 9$ are

$$0.0,\ 100\pi,\ 200\pi,\ 300\pi,\ 400\pi,\ \pm 500\pi,\ -400\pi,\ -300\pi,\ -200\pi,\ -100\pi$$