26 Windowing

In general, the sequence of a signal's sampling will not occupy a period that is related to the signal's fundamental frequency. In such a case, the corresponding Discrete Fourier Transform will not correspond to the Fourier series coefficients of the original signal.

The application of a DFT to a sequence of N samples of a signal taken at a regular period T considers a total time interval of only NT. Any part of the signal outside this interval is unknown.

The process of using only a limited interval, or window, of a signal is called *windowing*. In the time domain, windowing is multiplication by a function that is zero for all t outside the window interval.

A DFT of a sequence of N samples of a signal $f(t)$ is really a DFT of a signal $f_w(t)$, which is constructed with multiplication by a window function $g(t)$.

$$f_w(t) = f(t)g(t)$$

where $g(t) = 0$ for $t < 0$ and $t \geq P$.

In the frequency domain the windowing is a convolution of the Fourier transforms of $f(t)$ and $g(t)$.

$$F_w(\omega) = F(\omega) * G(\omega)$$

The DFT assumes that the sequence $f_w[k]$ is generated from the periodic signal

$$\sum_{n=-\infty}^{\infty} f_w(t - nP).$$

The DFT then generates the sequence $F_w[m] = F_w(m\omega_o)/T$.

Note that if $f(t) = 1$, then $F_w(\omega) = G(\omega)$.

26.1 Boxcar Window

The simplest window is the *boxcar*, defined in the window interval by $g(t) = 1$. That is, simply using values of the sampled signal within the window period.

The Fourier transform pair describing this window is

$$g(t) = \begin{cases} 1 & 0 \le t < P \\ 0 & \text{otherwise} \end{cases}$$

$$G(\omega) = 2\frac{\sin \omega P/2}{\omega} e^{-j\omega P/2}$$

If $f(t) = 1$, then $F_w(\omega) = G(\omega)$, and

$$F_w[m] = P(\sin(m\pi)/\pi m)e^{-jm\pi} = \{P,\ 0,\ 0,\ \ldots\}.$$

In this case the spectrum of the sampled signal is not altered by the window.

How windowing works

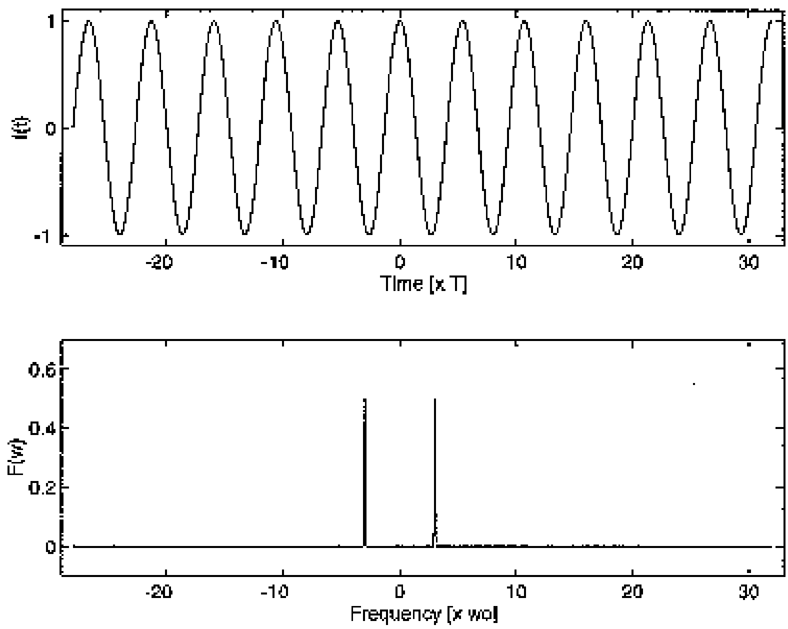
Consider a DFT of N samples of continuous-time signal taken at a sampling interval of T. Let the signal have a frequency that is an integer multiple of $\omega_o = 2\pi/T$.

For example,

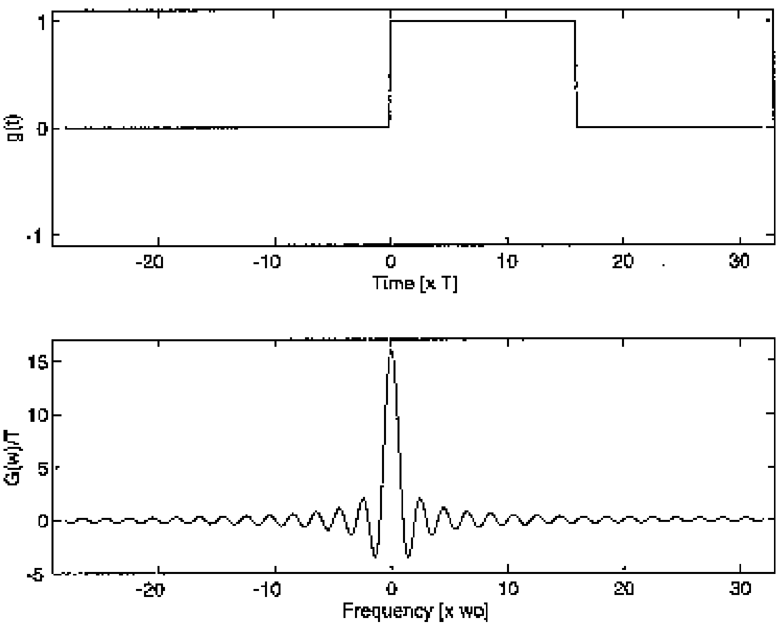
$$f(t) = \cos(6\pi t/T),$$

with Fourier transform.

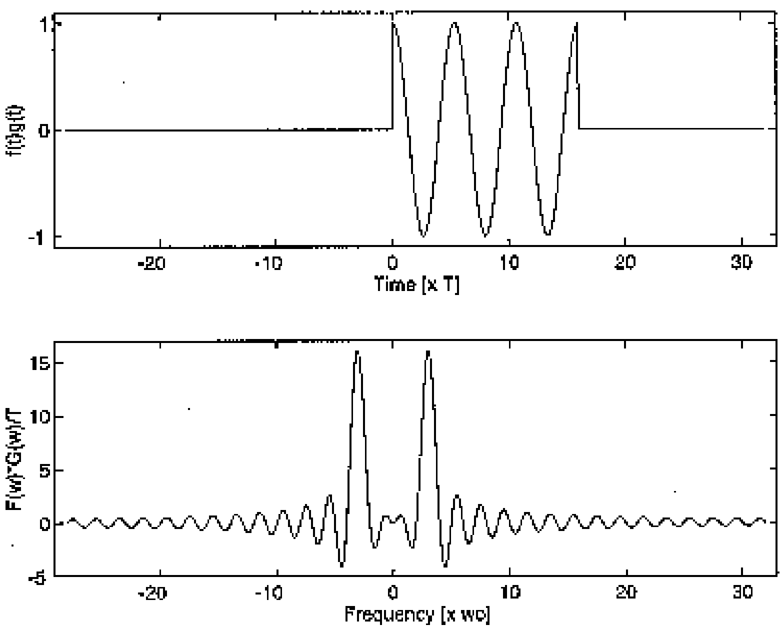
$$F(\omega) = 0.5\delta(\omega + 3\omega_o) + 0.5\delta(\omega - 3\omega_o).$$



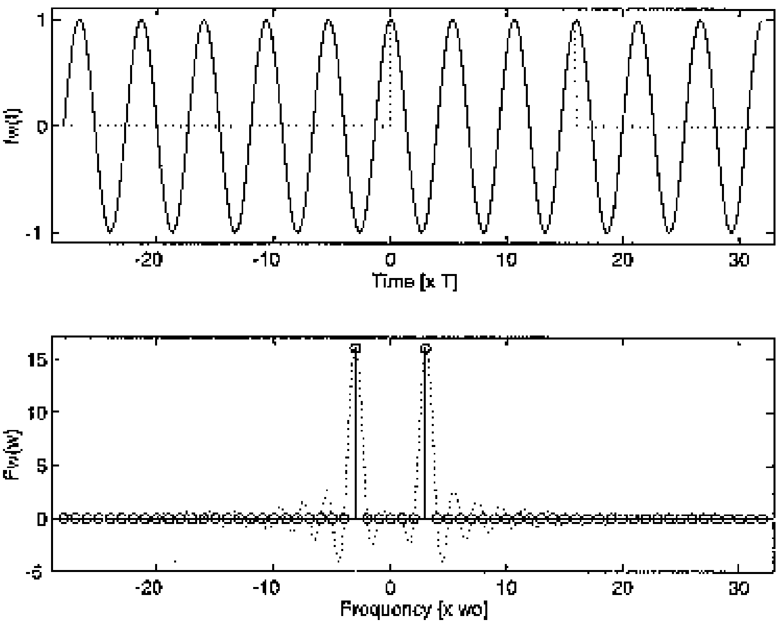
A DFT is a transform of one period of a signal. The single period of the signal to be transformed is constructed by multiplying $f(t)$ by the window function $g(t)$. A time and frequency domain plot of a boxcar window that selects the interval $0 < t \leq 16T$ is shown in the next diagram. (Note that the time-shift factor, $e^{-j\pi/8}$, has been ignored because it does not affect the magnitude.)



The windowed signal is the product of $f(t)$ and $g(t)$, which is a convolution in the frequency domain.



As far as the DFT is concerned the time function is periodic, so the frequency domain consists of impulses at multiples of ω_0. However, the original continuous-time signal has a frequency that is a multiple of ω_0. Thus the impulses in the frequency domain are coincident with zero values except at the frequency of the signal.



26.1.1 Spectral Leakage

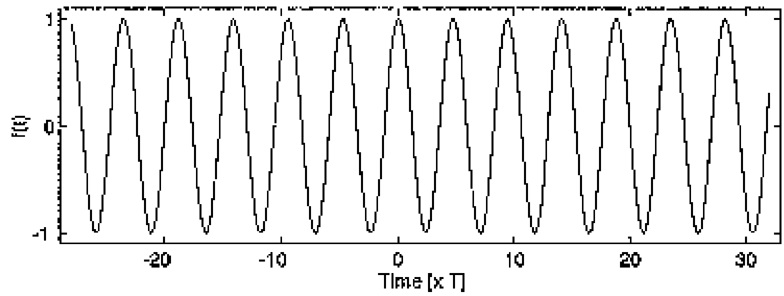
A problem arises if the continuous-time signal contains a frequency that is not a an integer multiple of $\omega_o = 2\pi/T$.

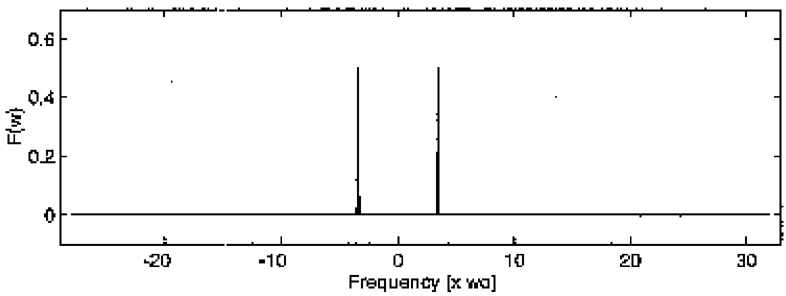
For example

$$f(t) = \cos(6.8\pi t/T),$$

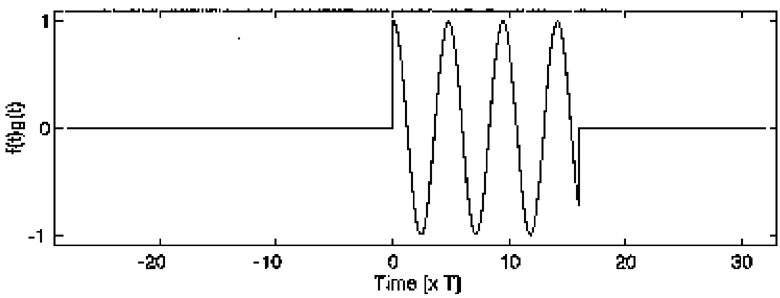
which has a Fourier transform of

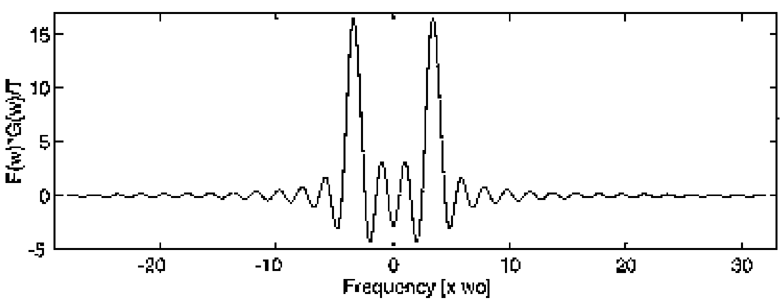
$$F(\omega) = 0.5\delta(\omega + 3.4\omega_o) + 0.5\delta(\omega 3.4\omega_o).$$



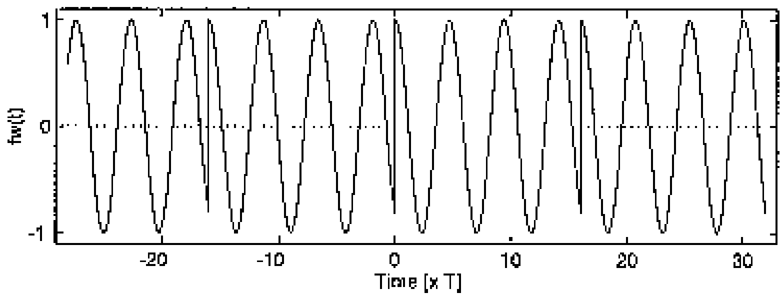


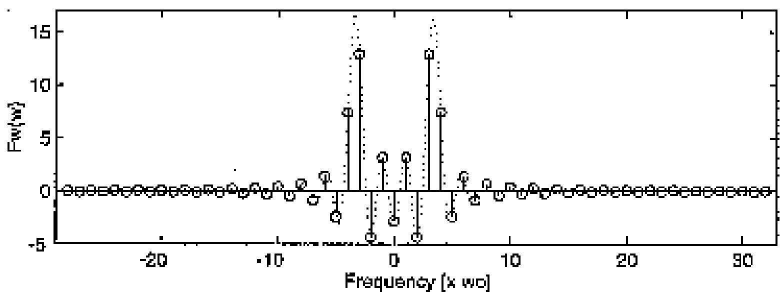
The windowed signal, which is the product of $f(t)$ and $g(t)$, and a convolution in the frequency domain, appears as follows.



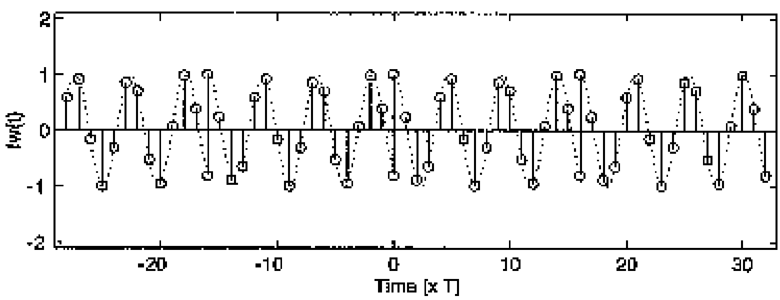


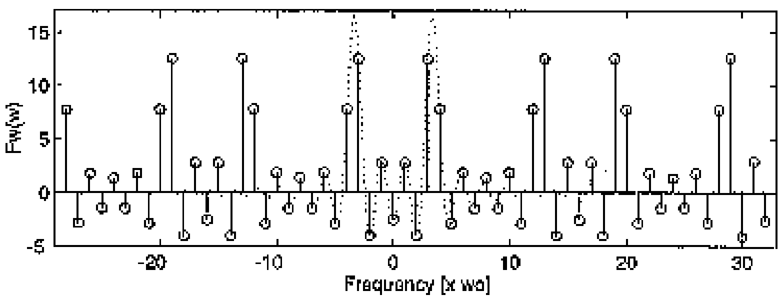
When the windowed time function is assumed to be periodic, it is quite different from the original. The impulses in the frequency domain are no longer coincident with zero values. These extra terms in the frequency domain are called *spectral leakage*. The *leakage* shows frequency components that are not in the original signal. They are an artifact arising because only a finite amount of the signal was analysed.





Note for completeness, in the context of the DFT, that the time-domain signal is also sampled. Thus the frequency domain becomes a periodic signal as shown below.





26.2 Raised-cosine Window

Spectral leakage can be managed by using special window functions. A simple, but effective, window is the *raised-cosine window*, which is defined by the following Fourier transform pair:

$$
\begin{aligned}
g(t) &= \begin{cases} 1 + \cos(2\pi t/P) & 0 \le t < P \\ 0 & \text{otherwise} \end{cases} \\
G(\omega) &= \left(\frac{\sin(\omega P/2 + \pi)}{\omega P/2 + \pi} + 2\frac{\sin \omega P/2}{\omega} + \frac{\sin(\omega P/2 - \pi)}{\omega P/2 - \pi} \right) e^{-j\omega P/2}
\end{aligned}
$$

If $f(t) = 1$, then $F_w(\omega) = G(\omega)$, and

$$F_\omega[m] = \{P,\ P/2,\ 0,\ \ldots,\ 0,\ P/2\}.$$

In this case the sampled spectrum has leakage at $m = 1$ and $m = N - 1$, but is zero otherwise.

To understand why the raised-cosine window is better, consider the signal that exhibited spectral leakage in the previous example.

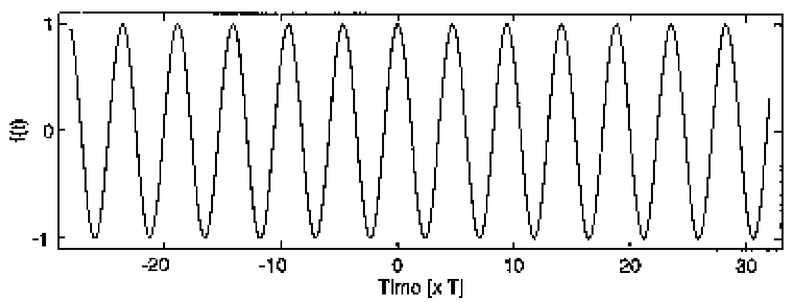
That is, let

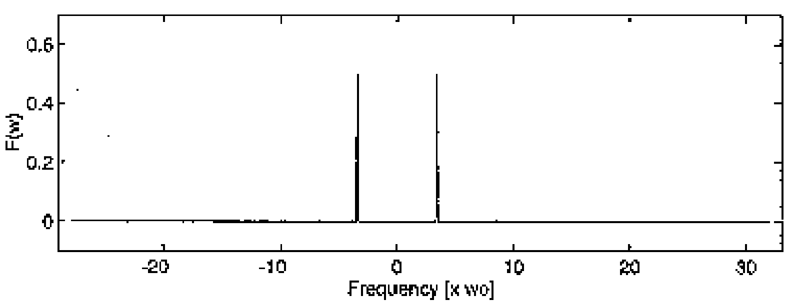
$$f(t) = \cos(6.8\pi t/T),$$

which has a Fourier transform of

$$F(\omega) = 0.5\delta(\omega + 3.4\omega_o) + 0.5\delta(\omega 3.4\omega_o)$$

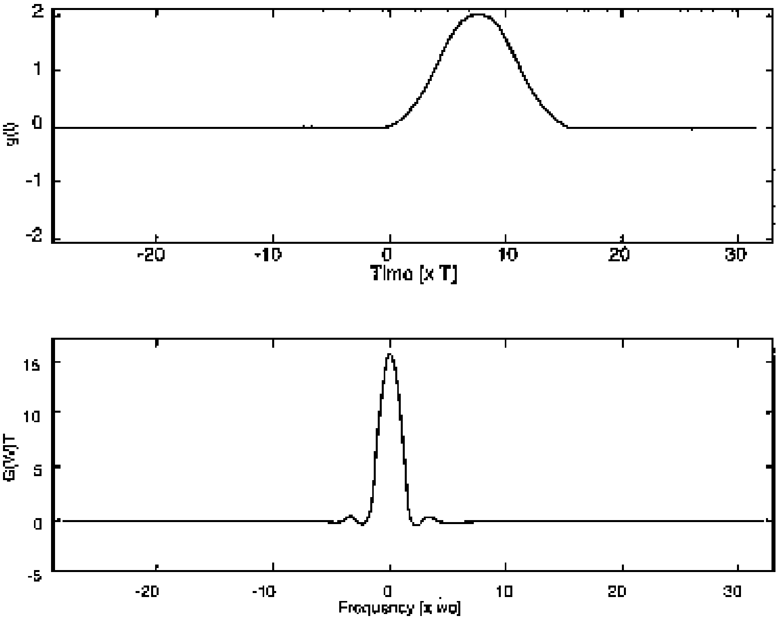
.



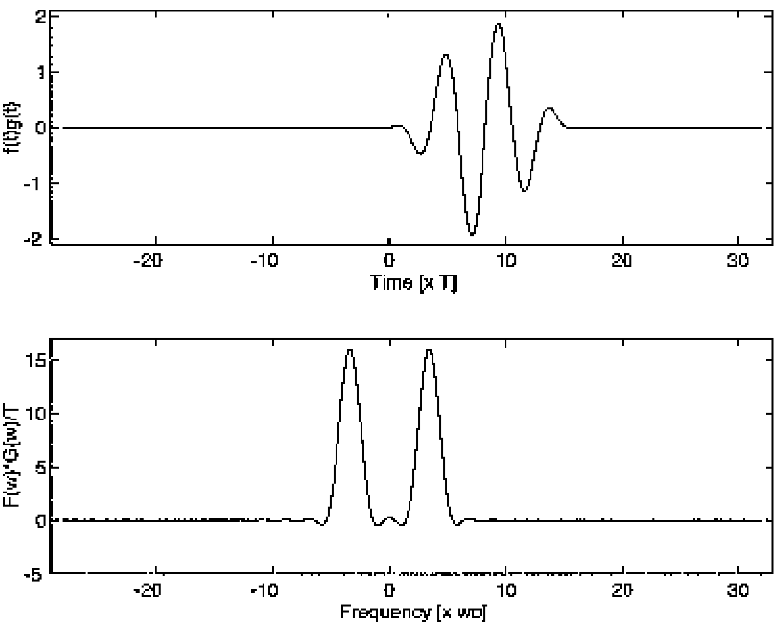


The time and frequency domain plot of a raised-cosine window that will select the interval $0 \le t < 16T$ is shown in the next diagram. (Note again that the time-shift factor, $e^{-j/8}$, has been ignored because it does not affect the magnitude.)

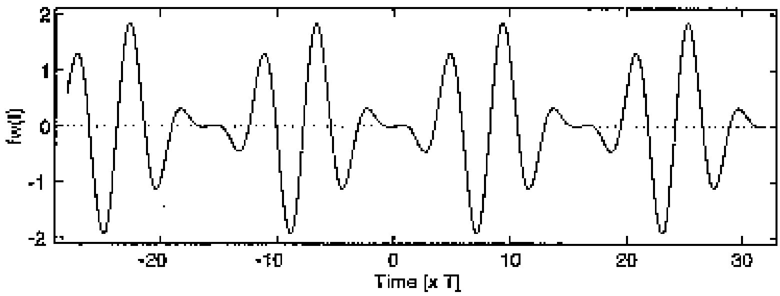
The important feature of the window is that the spectrum is near zero except near $\omega = 0$.

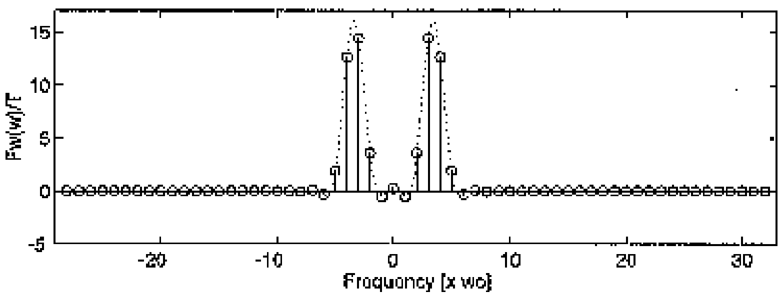


The windowed signal, which is the product of $f(t)$ and $g(t)$, and a convolution in the frequency domain, appears as follows.



When this windowed time function is extended to be periodic, it does not have a discontinuous step as with the boxcar window. The impulses in the frequency domain that are away from the original frequencies are now coincident with near zero values. Thus the *spectral leakage* is confined, or controlled, to a region near the original frequency components.





In the context of the DFT, that the time-domain signal is also sampled. Thus the frequency domain becomes a periodic signal as shown below.